Polar coordinates and double integrals

Old, compressed version of topic 32 notes.

Polar Coordinates
\[ x = r \cos \theta, \quad y = r \sin \theta, \quad r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}(y/x). \]

(\theta is tricky, \( \tan^{-1} \) is in quotes to indicate you need to pick the correct quadrant. Use the picture –see example below.)

Example: (more than one way to represent any point) (The \( x,y \) coordinates are at the top of each column and various \( r,\theta \) representations are below.)

\[
\begin{align*}
(x, y) & \quad (r, \theta) & \quad (r, \theta) & \quad (r, \theta) \\
(1, 0) & \quad (1, 0) & \quad (0, \pi/2) & \quad (2, 0) \\
(0, 1) & \quad (1, \pi/2) & \quad (\sqrt{2}, \pi/4) & \quad (\sqrt{2}, 3\pi/4) \\
(2, 0) & \quad (2, 0) & \quad (\sqrt{2}, 5\pi/4) & \quad (0, \pi/2) \\
(1, 1) & \quad (\sqrt{2}, \pi/4) & \quad (0, -7.2) & \quad (0, 0) \\
(-1, 1) & \quad (\sqrt{2}, 3\pi/4) & \quad (0, \pi/2) & \quad (0, \pi/2) \\
(-1, -1) & \quad (\sqrt{2}, 5\pi/4) & \quad (0, \pi/2) & \quad (0, \pi/2) \\
(0, 0) & \quad (0, \pi/2) & \quad (0, \pi/2) & \quad (0, \pi/2) \\
\end{align*}
\]

See the next pages for examples of graphs of functions in polar coordinates.

Double integral: \( \int \int_R f(x, y) \, dA \)

\( dA \) in polar coordinates: \( \Delta A \approx r \Delta \theta \Delta r \Rightarrow dA = r \, d\theta \, dr = r \, dr \, d\theta \)

Example: Find the mass of the region \( R \) shown if it has density \( \delta(x, y) = xy \)

In polar coordinates: \( \delta = r^2 \cos \theta \sin \theta \).

Limits of integration: (radial lines sweep out \( R \)):

inner (fix \( \theta \)): \( 0 < r < 2 \), outer: \( 0 < \theta < \pi/3 \).

\[ \Rightarrow \text{Mass } M = \int \int_R \delta(x, y) \, dA = \int_{\theta=0}^{\pi/3} \int_{r=0}^{2} r^2 \cos \theta \sin \theta \, r \, d\theta \, dr \]

Inner: \( \int_0^2 r^3 \cos \theta \sin \theta \, dr = \frac{r^4}{4} \cos \theta \sin \theta \Bigg|_0^2 = 4 \cos \theta \sin \theta \)

Outer: \( M = \int_0^{\pi/3} 4 \cos \theta \sin \theta \, d\theta = 2 \sin^2 \theta \Bigg|_0^{\pi/3} = \frac{3}{2} \).

Example: \( I = \int_1^2 \int_0^{\pi/2 \sqrt{x^2+y^2}} \frac{1}{(x^2+y^2)^{3/2}} \, dy \, dx \)
Draw the region.

Limits in polar coordinates:
inner (fix $\theta$): $\sec \theta < r < 2 \sec \theta$, outer: $0 < \theta < \pi/4$.

$\Rightarrow I = \int_{\theta=0}^{\pi/4} \int_{r=\sec \theta}^{2 \sec \theta} \frac{1}{r^3} r \, dr \, d\theta$.

Inner:
$\int_{\sec \theta}^{2 \sec \theta} \frac{1}{r^2} \, dr = -\frac{1}{r} \bigg|_{\sec \theta}^{2 \sec \theta} = 2 \sec \theta = \frac{1}{2} \cos \theta$.

Outer:
$I = \int_{0}^{\pi/4} \frac{1}{2} \cos \theta \, d\theta = \frac{1}{2} \sin \theta \bigg|_{0}^{\pi/4} = \frac{\sqrt{2}}{4}$.

**Example:** Find the volume of the region above the $xy$-plane and below the graph of $z = 1 - x^2 - y^2$.

You should draw a picture of this.

In polar coordinates we have $z = 1 - r^2$ and we want the volume under the graph and above the inside of the unit disk.

$\Rightarrow \text{volume } V = \int_{0}^{2\pi} \int_{0}^{1} (1 - r^2) \, r \, dr \, d\theta$.

Inner integral:
$\int_{0}^{1} (1 - r^2) \, r \, dr = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

Outer integral:
$V = \int_{0}^{2\pi} \frac{1}{4} \, d\theta = \frac{\pi}{2}$.

**Gallery of polar graphs** ($r = f(\theta)$)

A point $P$ is on the graph if any representation of $P$ satisfies the equation.

**Examples:**

Ray: $\theta = \pi/3$
$\Rightarrow r = 2$

Circle centered on 0:
$\Rightarrow r = 2 \sec \theta$

Vertical line $x = 2 \Leftrightarrow r = 2$.

Horizontal line $y = 2 \Leftrightarrow r = 2/\sin \theta$. 

\[ \begin{align*}
\text{Ray: } &\theta = \pi/3 \\
\text{Circle centered on 0: } &r = 2 \\
\text{Vertical line } &x = 2 \\
\text{Horizontal line } &y = 2
\end{align*} \]
Example: Describe the graph of \( r = 2a \cos \theta \).

First plot as usual.

(On \( \theta = [0, 2\pi] \) graph goes around twice)

Then determine the graph analytically:

\[
\begin{align*}
    r^2 &= 2ar \cos \theta = 2ax. \\
    \Rightarrow x^2 + y^2 &= 2ax. \\
    \Rightarrow (x - a)^2 + y^2 &= a^2.
\end{align*}
\]

This is a circle or radius \( a \) centered at \((a, 0)\).

Note: the nicest range for \( \theta \) is \(-\pi/2 \leq \theta \leq \pi/2\).

Warning: We can use \( r \) negative for plotting. You should never use it in integration.

In integration it is better to make use of symmetry and only integrate over regions where \( r \) is positive.

Cardiod: \( r = a(1 + \cos \theta) \)

Limaçon: \( r = a(1+b \cos \theta) \ (b > 1) \)

Lemniscate: \( r^2 = 2a^2 \cos 2\theta \)

Four leaved rose: \( r = a \sin 2\theta \)
**Ellipse** (we’ll probably skip this in class):

We will derive a formula from the geometric definition.

Geometry: ellipse = all points so that sum of lengths of distance from the two foci is a given constant.

(To draw: take loop of string and two thumbtacks...)

⇒ ellipse is all points such that $d_1 + d_2 = C$ ($C$ given constant).

Put foci at $(±a, 0)$.

**Law of cosines:**

\[
\begin{align*}
  d_1^2 &= r^2 + a^2 - 2ar \cos \theta \\
  d_2^2 &= r^2 + a^2 - 2ar \cos(\pi - \theta) = r^2 + a^2 + 2ar \cos \theta
\end{align*}
\]

⇒ $\sqrt{r^2 + a^2 - 2ar \cos \theta} + \sqrt{r^2 + a^2 + 2ar \cos \theta} = C$

**Algebra:** (I won’t do this in class)

\[
\begin{align*}
  2(r^2 + a^2) + 2\sqrt{(r^2 + a^2)^2 - 4a^2r^2 \cos^2 \theta} &= C^2 \\
  C^2 - 2(r^2 + a^2) &= 2\sqrt{(r^2 + a^2)^2 - 4a^2r^2 \cos^2 \theta} \\
  C^4 - 4C^2(r^2 + a^2) + 4(r^2 + a^2)^2 &= 4((r^2 + a^2)^2 - 4a^2r^2 \cos^2 \theta) \\
  -16a^2r^2 \cos^2 \theta + 4C^2r^2 &= C^2(C^2 - 4a^2) \\
  4r^2(C^2 - 4a^2 \cos^2 \theta) &= C^2(C^2 - 4a^2)
\end{align*}
\]

(Note, this implies we must have $C > 2a$, as is obvious from the geometric definition of the ellipse.)