49 More Stokes theorem

Review: We have learned the following.
1. Simply connected: A volume \( D \) is simply connected if every closed curve in \( D \) can be shrunk to a point without leaving \( D \). We stated the following theorem.

   **Theorem:** If \( D \) is simply connected then every closed curve in \( D \) is the boundary of some two-sided (orientable) surface in \( D \).

2. **Theorem:** (A) If \( F = \nabla f \) then \( \text{curl} F = 0 \).

   (B) If \( F \) is continuously differentiable on all \( x, y, z \) and \( \text{curl} F = 0 \) then \( F = \nabla f \) for some potential function \( f \).

   (B’): If \( D \) is a simply connected volume (defined below), \( F \) is continuously differentiable in \( D \) and \( \text{curl} F = 0 \) then in \( D \), \( F = \nabla f \) for some potential function \( f \).

3. **Stokes’ Theorem:** Let \( F \) be a vector field. Let \( S \) be an oriented surface with boundary \( C \), compatibly oriented. If \( F \) is continuously differentiable on \( S \) then

   \[
   \oint_C F \cdot dr = \iint_S \text{curl} F \cdot n \, dS = \iint_S \nabla \times F \cdot n \, dS.
   \]

   We see immediately that Stokes’ theorem implies (B) and (B’).

   **Proof:** If \( C \) is a closed curve in \( D \) then take \( S \) an orientable surface in \( D \) with boundary \( C \). By Stokes’ Theorem we have \( \oint_C F \cdot dr = \iint_S \text{curl} F \cdot n \, dS = 0 \). That is, the work integral is 0 around all closed loops, hence \( F \) is conservative.

**Fun with topology:** The theorem in statement 1 above is actually very deep and subtle. The surface \( S \) is called a Seifert surface after one of the people who proved the theorem. We’ll look at one example to see some of the complications.

**Example 1:** Cylinder and Mobius strip.

Cylinder: two-sided.
Can choose \( n \) pointing ’out’.
Boundary = \( C_1 + C_2 \).

Mobius strip: one-sided.
No way to choose \( n \) continuously.
Boundary = \( C \).

The cylinder \( S \) is orientable, it has 2 sides and we can choose \( n \) the ’outward’ normal. Its boundary consists of two curves \( C_1 + C_2 \). Stokes’ theorem works here for a vector field \( F \) (continuously differentiable on \( S \)):

\[
\oint_{C_1 + C_2} F \cdot dr = \iint_S \text{curl} F \cdot n \, dS.
\]
The Mobius strip $M$ is not orientable, it has only one side. Its boundary is one curve $C$. We can’t make a continuous choice of normals, so there is no way to apply Stokes’ theorem on $M$.

But statement 1 above says there is some orientable surface $S$ with $C$ as its boundary. The pictures show what it looks like.

Example 2: In section V14 of the supplementary notes a more complicated example is given. Instead of 1 twist we put 3 twists in our strip of paper before connected the ends. This surface is also a Mobius strip and is not orientable. Its boundary is a single curve called the trefoil knot.

Statement 1 says there is an orientable surface with $C$ as its boundary. We won’t describe it, but there is a well defined algorithm for producing such a surface for any curve.

Extended Stokes’ Theorem: If $S$ is an oriented surface with boundary consisting of 2 (or more curves) then Stokes’ theorem still applies provided the curves and surface are compatibly oriented.

Example 3: \[ \oint_{C_1 + C_2} \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F} \cdot \mathbf{n} \, dS. \]

Example 4: Sometimes we orient the curves differently and work carefully with the signs \[ \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} - \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F} \cdot \mathbf{n} \, dS. \]

Example 5: Show $\mathbf{F} = \rho^n (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ is a gradient field.

answer: An easy calculation shows $\text{curl}\mathbf{F} = 0$. (As usual, use $\frac{\partial \rho}{\partial x} = \frac{x}{\rho}$, etc.) Since $\mathbf{F}$ is defined on space minus the origin, which is simply connected. Thus, $\text{curl}\mathbf{F} = 0 \Rightarrow \mathbf{F}$ is a gradient field.

Example 6: Let $\mathbf{F} = \langle 2xz + 2y, 2yz + 2yx, x^2 + y^2 + z^2 \rangle$. Take $C_1$ and $C_2$ two curves going around the circular cylinder of radius $a$ as shown. Show $\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

answer: We easily compute $\text{curl}\mathbf{F} = (2y - 2)\mathbf{k} \Rightarrow \text{curl}\mathbf{F} \cdot \mathbf{n} = 0$, where
\( \mathbf{n} \) is the normal to the cylinder. Let \( S \) be the part of the cylinder between \( C_1 \) and \( C_2 \) then Stokes’ theorem says

\[
\oint_{C_1 - C_2} \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl} \mathbf{F} \cdot \mathbf{n} \, dS = 0. \quad \Rightarrow \quad \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}. \quad \text{QED}
\]