50 Applications and interpretation of Stokes theorem

Read section V15 in the supplementary notes.

Divergence and Stokes’ Theorems in del notation

Divergence Theorem: \[ \int_S \mathbf{F} \cdot \mathbf{n} \, dS = \int_D \text{div} \mathbf{F} \, dA = \int_D \nabla \cdot \mathbf{F} \, dA. \]

Stokes’ Theorem: \[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_S \text{curl} \mathbf{F} \cdot \mathbf{n} \, dS = \int_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, dS. \]

Maxwell’s Equations

- \( S \): closed surface
- \( D \): interior of \( S \)
- \( C_1 \): closed curve
- \( S_1 \): surface capping \( C_1 \)
- \( \mathbf{E} \): electric field
- \( \mathbf{B} \): magnetic field
- \( Q \): charge inside \( S \)
- \( \rho \): charge density inside \( S \)
- \( \mathbf{J} \): current density

\( k \) and \( k_1 \) are constants depending on units.

Integral form of Maxwell’s equations

1') \[ \int_S \mathbf{E} \cdot \mathbf{n} \, dS = 4\pi kQ \] Gauss-Coulomb Law.

2') \[ \int_S \mathbf{B} \cdot \mathbf{n} \, dS = 0 \] Gauss’ law of magnetism.

3') \[ \oint_{C_1} \mathbf{E} \cdot d\mathbf{r} = -\frac{d}{dt} \int_{S_1} \mathbf{B} \cdot \mathbf{n} \, dS \] Faraday’s law.

4') \[ \oint_{C_1} \mathbf{B} \cdot d\mathbf{r} = k_1 \int_{S_1} \mathbf{J} \cdot \mathbf{n} \, dS + k \frac{d}{dt} \int_{S_1} \mathbf{E} \cdot \mathbf{n} \, dS \] Ampere’s law.

Differential form of Maxwell’s equations (equivalent to the integral form).

1) \( \nabla \cdot \mathbf{E} = 4\pi k\rho \) Gauss-Coulomb Law.

2) \( \nabla \cdot \mathbf{B} = 0 \) Gauss’ law of magnetism.

3) \( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \) Faraday’s law.

4) \( \nabla \times \mathbf{B} = k_1 \mathbf{J} + \frac{k_1}{k} \frac{\partial \mathbf{E}}{\partial t} \) Ampere’s law.

In words these statements say the following:

1,1’) Flux of \( \mathbf{E} \) through \( S = 4\pi k \times \text{charge enclosed} \).

2,2’) There is no magnetic monopole. (No net magnetic charge enclosed.)
3,3’) A changing magnetic field induces an electric field.
4,4’) Magnetic fields are induced by either a current or a changing electric field.

We discussed Gauss’ law in topic 46 with respect to gravitation. Here’s a quick recap for electricity.

For a charge $q$ at the origin the electric field is $E_q = k \frac{\langle x, y, z \rangle}{\rho^3}$ (here $\rho$ is distance from the origin, not charge density).

By direct computation the flux of $E_q$ through a sphere centered on the origin is $4\pi kq$.

Everywhere but the origin $\nabla \cdot E_q = 0$, so the extended divergence theorem gives flux of $E_q$ through $S = \begin{cases} 4\pi kq & \text{if } S \text{ goes around } q \\ 0 & \text{otherwise.} \end{cases}$

In general, if $q$ is placed anywhere else the same formula holds.

Now suppose $q_1, q_2, \ldots, q_n$ are charges with fields $E_1, E_2, \ldots, E_n$. Let $E = E_1 + \ldots + E_n$ be the net electric field. The net flux through $S$ is the sum of the flux contributed by each field $E_j$. That is, the flux of $E$ through $S = 4\pi k \cdot \times \text{(total charge inside } S)$. For a continuous charge distribution we replace sums by integrals and find the same thing. This proves (1’).

Before showing (1) and (1’) are equivalent we state a very reasonable theorem.

**Theorem:** Assume $h(x, y, z)$ is a continuous function and $\iiint_D h \, dV = 0$ for every volume $D$ then $h(x, y, z) = 0$.

**Proof:** We argue by contradiction. Suppose for some point $(x_0, y_0, z_0)$ we have $h(x_0, y_0, z_0) > 0$. Since $h$ is continuous we can put a small ball $D$ around $(x_0, y_0, z_0)$ such that $h(x, y, z) > 0$ throughout $D$. This implies $\iiint_D h \, dV > 0$, which contradicts our assumption. Therefore it can’t be positive. Likewise it can’t be negative. QED

**Corollary:** Assume $f(x, y, z)$ and $g(x, y, z)$ are continuous and $\iiint_D f \, dV = \iiint_D g \, dV$ for every volume $D$ then $f(x, y, z) = g(x, y, z)$.

**Proof:** Let $h = f - g$ and apply the theorem.

Now we show (1) and (1’) are equivalent.

The divergence theorem implies $\iint_S E \cdot n \, dS = \iiint_D \nabla \cdot E \, dV$.

The definition of density says: $4\pi kQ = 4\pi k \iint_D \rho \, dV$.

Thus, $\iint_S E \cdot n \, dS = 4\pi kQ \iff \iint_D \nabla \cdot E \, dV = 4\pi k \iint_D \rho \, dV$.

Since this is true for every $D$ the above corollary says it is equivalent to $\nabla \cdot E = 4\pi k \rho$. QED

Showing (2), (3) and (4) are equivalent to (2’), (3’) and (4’) respectively is similar.