9 Integration: partial fractions

Example 9.1. Algebra shows that \( \frac{4}{x-3} - \frac{1}{x-1} = \frac{3x-1}{x^2-4x+3} \).

Question: How do we go backwards?

answer: Suppose we are given \( \frac{3x-1}{x^2-4x+3} \), then by factoring the denominator we can write

\[
\frac{3x-1}{x^2-4x+3} = \frac{3x-1}{(x-3)(x-1)} = \frac{A}{x-3} + \frac{B}{x-1}.
\]

The goal now is to determine the values of \( A \) and \( B \).

The long method is to cross-multiply which gives

\[
3x-1 = A(x-1) + B(x-3).
\]

If we let \( x = 1 \) we see that \( B = -1 \).

Likewise, if \( x = 3 \) we see \( A = 4 \).

Coverup method: Doing the above without writing. This is very useful. You should read the supplementary notes §F. Here is the basic idea. The partial fractions decomposition is

\[
\frac{3x-1}{x^2-4x+3} = \frac{3x-1}{(x-3)(x-1)} = \frac{A}{x-3} + \frac{B}{x-1}.
\] (1)

Multiply both sides by \( x-3 \). This produces,

\[
\frac{3x-1}{x-1} = A + \frac{B(x-3)}{x-1}.
\] (2)

Now set \( x = 3 \) to find that \( 8/2 = A \), i.e. \( A = 4 \) (as before). Likewise, multiply both sides by \( x-1 \) to get:

\[
\frac{3x-1}{x-3} = \frac{A(x-1)}{x-3} + B
\]

Set \( x = 1 \) to find that \( 2/(-2) = B \), i.e. \( B = -1 \).

Why is this called coverup? Because you can perform the calculation without writing things out. Looking at equation 2 we see that to find \( A \) all we have to do with Equation 1 is ‘coverup’ the factor of \((x-3)\) in the denominator and substitute \( x = 3 \) into what is left.

Question: Why go backwards?

answer: Because the partial fraction terms are easy to integrate!

Example 9.2. Compute \( \int \frac{1}{x^2-3x+2} \, dx \).

answer: Partial fractions: \( \frac{1}{x^2-3x+2} = \frac{1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} \).
By coverup: \( A = -1, \ B = 1. \) Thus

\[
\text{integral } = \int \left( -\frac{1}{x-1} + \frac{1}{x-2} \right) \, dx = -\ln(x-1) + \ln(x-2) + C.
\]

Getting fancier:

**Repeated linear factors:**

**Example 9.3.** Write \( \frac{x^2-2x+2}{(x+2)^2(x-2)} \) in terms of partial fractions.

**answer:**

\[
\frac{x^2-2x+2}{(x+2)^2(x-2)} = \frac{A}{(x+2)^2} + \frac{B}{x+2} + \frac{C}{x-2} \quad \text{(One term for each power.)}
\]

Coverup gives \( A = \frac{-5}{2}, \ C = \frac{1}{8}. \)

To find \( B \) we cross multiply and solve:

\[
x^2-2x+2 = A(x-2)+B(x+2)(x-2)+C(x+2)^2 = (B+C)x^2+(A+4C)x+(-2A-4B+4C).
\]

When two polynomials are the same, the coefficients must be equal. So,

Coefficient of \( x^2 \): \( 1 = B + C. \) Since \( C \) is known, we get \( B = \frac{7}{8}. \)

Since we’ve found \( A, \ B \) and \( C \) there is no need to write down the other equations. We record them so you can check our answer is correct.

Coefficient of \( x \): \( -2 = A + 4C. \)

Coefficient of 1 (constant terms): \( 2 = -2A - 4B + 4C. \)

**Quadratic factor:**

**Example 9.4.** Write \( \frac{x^2+3}{(x^2+1)(x-1)} \) in terms of partial fractions.

**answer:** Since the \( x^2 + 1 \) term in the denominator doesn’t factor, it appears as is in the partial fractions decomposition. It’s numerator must be of the form \( Bx + C. \)

\[
\frac{x^2+3}{(x^2+1)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.
\]

Coverup gives \( A = 2. \) We can’t use coverup to find \( B \) or \( C, \) so we cross multiply and equate coefficients.

\[
x^2+3 = A(x^2+1) + (Bx + C)(x-1) = (A + B)x^2 + (C - B)x + (A - C).
\]

Coefficient of \( x^2 \): \( 1 = A + B, \) \( A \) is known, so \( B = -1. \)

Coefficient of \( x \): \( 0 = C - B, \) so \( C = -1. \)

We have all the unknowns, so we don’t need to equate constant terms.

**Example 9.5.** (Integration)

Compute the integral

\[
I = \int \frac{x^2+3}{(x^2+1)(x-1)} \, dx.
\]

**answer:** From the previous example we know the partial fraction decomposition of the
integrating. So,
\[ I = \int \frac{2}{x - 1} dx - \int \frac{x}{x^2 + 1} dx - \int \frac{1}{x^2 + 1} dx \]
\[ = 2 \ln |x - 1| - \frac{1}{2} \ln(x^2 + 1) - \tan^{-1} x + C. \]

(You should memorize \( \int \frac{1}{x^2 + 1} = \tan^{-1} x + C \).)

**Repeated Quadratic Factor:**

**Example 9.6.** Write \( \frac{2x + 1}{(x^2 + 1)^2(x - 1)} \) in terms of partial fractions.

**answer:** We need one term for each power of \((x^2 + 1)\). That is
\[
\frac{2x + 1}{(x^2 + 1)^2(x - 1)} = \frac{Ax + B}{(x^2 + 1)^2} + \frac{Cx + D}{x^2 + 1} + \frac{E}{x - 1}/
\]

Coverup gives \( E = 3/4 \).

We can’t get the other unknowns using coverup, so we cross multiply
\[
2x + 1 = (Ax + B)(x - 1) + (Cx + D)(x^2 + 1)(x - 1) + E(x^2 + 1)^2 = (C + E)x^4 + (D - C)x^3 + (A + C - D + 2E)x^2 + (-A + B - C)x + 1
\]

Equate coefficients:
Coefficient of \( x^4 \): \( 0 = C + E \). \( E \) is known, so \( C = -3/4 \)
Coefficient of \( x^3 \): \( 0 = D - C \) \( \Rightarrow D = -3/4 \)
Coefficient of \( x^2 \): \( 0 = A + C - D + 2E \) \( \Rightarrow A = -3/2 \)
Coefficient of \( x \): \( 2 = -A + B - C + D \) \( \Rightarrow B = 1/2 \)
Coefficient of 1: \( 1 = -B - D + E \) (Don’t need this, but it checks out.)

**Example 9.7.** (More integration)

Compute \( \int \frac{x}{(x^2 + 1)^2} dx \).

**answer:** Note: this is not a partial fractions problem, because this is already a typical partial fractions term.

Substitute \( u = x^2 + 1 \), \( du = 2x \) \( dx \). Thus,
\[
\text{integral} = \int \frac{1}{2u^2} du = -\frac{1}{2}u^{-1} + C = -\frac{1}{2}(x^2 + 1)^{-1} + C.
\]

**Long division**

For partial fractions the rational function must be proper, i.e. the degree of the numerator must be less than the degree of the denominator.

**Example 9.8.** Decompose \( \frac{x^3 + 2x + 1}{x^2 + x - 2} = \frac{x^3 + 2x + 1}{(x + 2)(x - 1)} \) using partial fractions.

**answer:** First use long division to write this as a polynomial plus a proper rational function.
\[
x^2 + x - 2 \div x - 1
\]
\[
x^3 + 2x + 1
\]
\[
x^3 + x^2 - 2x
\]
\[
x^3 + x^2 - 2x + 1
\]
\[
x^2 - x + 2
\]
\[
x^2 - x + 2
\]
\[
5x - 1
\]
This shows that
\[
\frac{x^3 + 2x + 1}{x^2 + x - 2} = x - 1 + \frac{5x - 1}{x^2 + x - 2}.
\]

Now proceed as always.

\[
\frac{5x - 1}{x^2 + x - 2} = \frac{5x - 1}{(x + 2)(x - 1)} = \frac{A}{x + 2} + \frac{B}{x - 1}.
\]

Coverup gives \(A = 11/3\) and \(B = 4/3\). So the integral in question is

\[
\int x - 1 + \frac{A}{x + 2} + \frac{B}{x - 1} = \frac{x^2}{2} - x + A \ln(|x + 2|) + B \ln(|x - 1|) + C.
\]