ES.1803: Complex Impedance and Phasors
Jeremy Orloff

Impedance: Generalizes Ohm’s law $V = IR$ to capacitors and inductors.
Recall: Two resistances $R_1$ and $R_2$ combine to give an equivalent resistance $R$.
For $R_1, R_2$ in series $R = R_1 + R_2$ and in parallel $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.

We are going to use the exponential response formula and complex arithmetic to understand the notions of complex impedance and phasor diagrams.

Simple circuit physics
The picture at right shows an inductor, capacitor and resistor in series with a driving voltage source.

$I(t)$ is the current in the circuit in amps.
$L$ is the inductance in henries.
$R$ is the resistance in ohms.
$C$ is the capacitance in farads.
$V_{in}$ is the input voltage to the circuit.
$Q(t)$ is the charge on the capacitor, so $I(t) = Q'(t)$.

From physics we get that the voltage drops across each of the circuit elements.

$$V_L = LI' = LQ'', \quad V_R = RI = RQ', \quad V_C = \frac{Q}{C}.$$ 

The amazing thing is that this and Kirchhoff’s voltage law (KVL) is all the physics we need to understand this circuit. The rest is linear CC DE’s and complex arithmetic.

Summary: We start with a summary of our results. Individual items will be explained below.
Compatible units: current = amps, voltage = volts, resistance = ohms,
inductance = henries, capacitance = farads.

Voltage drops: $LI'$, $RI$, $\frac{1}{C} Q$.

DEs: $LQ'' + RQ' + \frac{1}{C} Q = V_{in}$; $LI'' + RI' + \frac{1}{C} I = V_{in}'$.

Complex impedance (valid when $V_{in} = e^{\omega t}$): $\tilde{Z}_L = iL\omega$, $\tilde{Z}_R = R$, $\tilde{Z}_C = \frac{1}{iC\omega}$,

total impedance $\tilde{Z} = \tilde{Z}_R + \tilde{Z}_L + \tilde{Z}_C = R + i(\omega L - 1/(\omega C))$.

Complex Ohm’s Law: $\tilde{V}_{in} = \tilde{Z}\tilde{I}$, $\tilde{V}_L = \tilde{Z}_L\tilde{I}$, $\tilde{V}_R = \tilde{Z}_R\tilde{I}$, $\tilde{V}_C = \tilde{Z}_C\tilde{I}$.

Phasors: All the output voltages are plotted in the complex plane as a rigid set of vectors that rotate at frequency $\omega$. $V_R$ and $\tilde{I}$ point in the same direction, $\tilde{V}_L$ leads $\tilde{I}$ by $\pi/2$, $\tilde{V}_C$ lags $\tilde{I}$ by $\pi/2$. $\tilde{I}$ either leads or lags $\tilde{V}_{in}$ by $\phi = \text{atan2}(L\omega - 1/(C\omega), R)$. 

1
**Reactance and real impedance:**

Reactance \( \omega L + 1/(\omega C) \). \( \Rightarrow \hat{Z} = R + iS. \)

Real impedance \( |\hat{Z}| = \sqrt{R^2 + S^2} = \sqrt{\omega L + 1/(\omega C)^2}. \)

If \( V_{in} = E_0 \sin \omega t \) then from \( \hat{V}_{in} = \hat{Z}\hat{I} \) we get \( I = \frac{E_0}{|Z|} \sin(\omega t - \phi) \), with the phase angle \( \phi = \tan^{-1}(S/R) \).

I.e., the amplitude of the current is given by \( E_0/\text{real impedance} \). 

**Practical resonance:** Always at the natural frequency \( \omega_0 = 1/\sqrt{LC} \).

**Simple complex arithmetic fact**

You should be clear that in the complex plane multiplication by \( i \) is the same as rotation by \( \pi/2 \). Likewise division by \( i \) is the same as rotation by \( -\pi/2 \).

\[ z/\imath = -iz \]

**The basic differential equations**

KVL implies the total voltage drop around the circuit has to be 0. Which leads to the following second order CC linear DE.

\[ LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = V_{in}(t). \]

Differentiating this equation we can write

\[ LI''(t) + RI'(t) + \frac{1}{C}I(t) = V_{in}'(t). \]

**Complex Impedance**

Now complex arithmetic and the exponential response formula will allow us to understand all about phasors and impedance.

First, note that if we remove the inductor and capacitor the first DE is just Ohm's law, i.e. \( RQ' = RI = V_{in} \).

Now make the crucial assumption of sinusoidal input (alternating current):

\[ V_{in}(t) = V_0 \sin(\omega t). \]

Complexify the second equation: \( LI'' + RI' + \frac{1}{C}I = \hat{V}_{in}' = i\omega V_0 e^{i\omega t}, \quad I = \text{Im}(\hat{I}). \)

The exponential response formula gives the periodic solution: \( \hat{I} = \frac{i\omega V_0}{P(i\omega)} e^{i\omega t}. \)

A little algebra shows \( \frac{i\omega V_0}{P(i\omega)} = \frac{i\omega V_0}{-i\omega^2 + 1/C + Ri\omega} = \frac{V_0}{iL\omega + 1/(iC\omega) + R}. \)

Define \( \hat{Z} = iL\omega + \frac{1}{iC\omega} + R = \text{complex impedance} \) (which depends on the input frequency \( \omega \)).

We can now write the **complex version of Ohm’s law** (always assuming \( \hat{V}_{in} = V_0 e^{i\omega t} \)):

\[ \hat{I} = \frac{1}{\hat{Z}} \cdot \hat{V}_{in} \text{ or } \hat{V}_{in} = \hat{Z}\hat{I}. \]

Notice that we can associate a separate impedance to each element.
$Z_L = iL\omega, \quad Z_R = R, \quad Z_C = \frac{1}{iC\omega}$

We have seen that for a set of elements wired in series the total complex impedance is just the sum of the individual impedances. (Just like resistances in series.)

What’s more, using the voltage drops across each element we see they individually satisfy a Complex Ohm’s Law.

$\tilde{V}_L = L\tilde{I}' = L i\omega \tilde{I} = \tilde{Z}_L \tilde{I}, \quad \tilde{V}_R = R\tilde{I}, \quad \tilde{V}_C = \frac{1}{C} \tilde{Q} = \frac{1}{C} \int \tilde{I} = \frac{1}{iC\omega} \tilde{I} = \tilde{Z}_C \tilde{I}.$

**Impedance in parallel:** It is also true and easy to show that for circuit elements in parallel the complex impedances combine like resistors in parallel. That is, if impedances $\tilde{Z}_1$ and $\tilde{Z}_2$ are in parallel then the total impedance of the pair, call it $\tilde{Z}$, satisfies $\frac{1}{\tilde{Z}} = \frac{1}{\tilde{Z}_1} + \frac{1}{\tilde{Z}_2}$.

To see this we use the complex Ohm’s law, KVL and Kirchhoff’s current law (KCL). They imply

$\tilde{I} = \tilde{I}_1 + \tilde{I}_2, \quad \tilde{V} = \tilde{Z}_1 \tilde{I}_1, \quad \tilde{V} = \tilde{Z}_2 \tilde{I}_2.$

$\Rightarrow \tilde{I} = \frac{\tilde{V}}{Z_1} + \frac{\tilde{V}}{Z_2} = \tilde{V} \left( \frac{1}{Z_1} + \frac{1}{Z_2} \right)$

$\Rightarrow \tilde{V} = \frac{1}{1/Z_1 + 1/Z_2} \tilde{I}.$ QED

**Amplitude-phase form**

First rewrite $\tilde{Z} = iL\omega + \frac{1}{iC\omega} + R = i(L\omega - \frac{1}{C\omega}) + R = iS + R.$

Here, $S = L\omega - 1/(C\omega)$ is the reactance. Note $S = 0$ when $\omega^2 = 1/(LC)$.

In amplitude phase form $\tilde{Z} = re^{i\phi}$, where $r = \sqrt{S^2 + R^2}$ and $\phi = \text{atan2}(S,R)$.

Notice the sign of $\phi$ depends on the sign of $S = L\omega - 1/C\omega$ and also that $\phi$ is between $-\pi/2$ and $\pi/2$.

Thus $\tilde{I} = \frac{V_0}{\sqrt{S^2 + R^2}} e^{i(\omega t - \phi)} = \frac{V_0}{\sqrt{(L\omega - 1/C\omega)^2 + R^2}} e^{i(\omega t - \phi)}.$

The term $\sqrt{S^2 + R^2} = |\tilde{Z}| = \sqrt{(L\omega - 1/C\omega)^2 + R^2}$ is called the real impedance.

Taking imaginary parts: $I|\tilde{Z}| = V_0 \sin(\omega t - \phi)$. Which is like Ohm’s law, except with a phase shift.

**Phasors** (phasor just means $e^{i\omega t}$)

Note: the entire picture is rotating at frequency $\omega$ and the real values of each of the voltages are given by the $y$ coordinate (the imaginary part) of their respective phasors.

Note: the phasor $\tilde{V}_R$ is $\phi$ behind $\tilde{V}_m$ (if $\phi$ is negative then $\tilde{V}_R$ is ahead of $\tilde{V}_m$).

Note: the phasors $\tilde{V}_L$ and $\tilde{V}_C$ are respectively $\pi/2$ ahead and $\pi/2$ behind $\tilde{V}_R$.

In class we will look at the lovely SeriesLRC applet.
Amplitude response and practical resonance

Natural frequency \( w_0 = 1/\sqrt{LC} \) (Clearly when \( S = 0 \).)

Practical Resonance: Always at the natural frequency \( \omega_0 = 1/\sqrt{LC} \) (This is easy to see in the amplitude-phase form of \( \tilde{I} \), since \( \tilde{I} \) is clearly maximized when the term \( (L\omega - 1/C\omega)^2 = 0 \).)

I.e., practical resonance is when \( \tilde{Z}_L + \tilde{Z}_C = 0 \Rightarrow iL\omega - i/C\omega = 0 \Rightarrow \tilde{Z} = R, \tilde{I} = \frac{V_0}{R} e^{i\omega t}. \)

In the phasor picture, at practical resonance \( \tilde{V}_\text{in}, \tilde{I} \) and \( \tilde{V}_R \) all line up, i.e., lag is 0 and \( \tilde{V}_R = \tilde{V}_\text{in} \).

This is one case where the corresponding sinusoidal graphs of the real voltages are neat enough to give a nice picture: the graph of \( V_R \) is exactly in phase with \( V_\text{in} \); \( V_L \) and \( V_C \) have the same magnitude and are 180° out of phase; increasing \( R \) doesn’t change \( V_R \), but decreases the amplitude of \( V_L \) and \( V_C \).

The applet 'Series LRC Circuit' (link is given below) shows all this beautifully.

Applet A nice d’AIMP applet showing all of this is at [http://mathlets.org/mathlets/series-rlc-circuit/](http://mathlets.org/mathlets/series-rlc-circuit/)

Suggested applet exercise Set it to show you all four voltages and the current \( I \). Set \( L = 500 \text{ mH}, C = 100 \mu F, R = 250 \text{ ohms} \).

Compute the resonant frequency of the system.

Move \( \omega \) to the resonant frequency, watch the phasors and the sinusoidal plots as you do this.

With \( \omega \) set at \( \omega_0 \) watch the amplitudes of the 3 output voltages and the output current as \( R \) increases. Explain everything you see in terms of the complex Ohm’s laws. (And the exponential response formula solution for \( \tilde{I} \).)