18.03: Existence and Uniqueness Theorem
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**Theorem** (existence and uniqueness)

Suppose \( f(t, y) \) and \( \frac{\partial f}{\partial y}(t, y) \) are continuous on a rectangle \( D \) as shown. Then we can choose a smaller rectangle \( R \) (as shown) so that the IVP

\[
\frac{dy}{dt} = f(t, y(t)), \quad y(t_0) = y_0
\]

has a unique solution defined on \([t_0 - a, t_0 + a]\) whose graph is entirely inside \( R \).

The proof proceeds in a series of steps. Some of these steps are technical – I’ll try to give a sense of why they are true. The key steps are the definition of the contraction map \( T \) (step (3)) and the use of \( T \) in Picard iteration (step (8)).

(1) Let \( M = \max_D |f(t, y)| \) and \( L = \max_D |\frac{\partial f}{\partial y}(t, y)| \).

The mean value theorem \( \Rightarrow f(t, y_2) - f(t, y_1) = \frac{\partial f}{\partial y}(t, c)(y_2 - y_1) \) (for some \( c \) between \( y_1 \) and \( y_2 \)). \( \Rightarrow |f(t, y_2) - f(t, y_1)| < L|y_2 - y_1| \) (Lipschitz condition).

(2) **Choosing the rectangle** \( R \)

Choose \( a < \min\left(\frac{b}{M}, \frac{1}{2L}\right) \). We will use this in steps (3) and (5).

(3) **The operator** \( T \)

Let \( Y \) be the space of all functions \( y \) which are continuous on \([t_0 - a, t_0 + a]\) and whose graph is entirely inside \( R \). For any \( y \in Y \) define

\[
Ty = z(t) = y_0 + \int_{t_0}^{t} f(s, y(s)) \, ds.
\]

We note a number of easy facts about \( T \).

(a) \( Ty = z(t) \) is well defined on \([t_0 - a, t_0 + a]\). (proof: \( (s, y(s)) \) is in \( R \), so the integrand \( f(s, y(s)) \) is defined and continuous.)
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(b) \( z(t) \) is continuous. (proof: trivial since both \( y \) and \( f \) are continuous.)

(c) The graph of \( z(t) \) is entirely in \( R \).

proof: \( |z(t) - y_0| = \left| \int_{t_0}^{t} f(s, y(s)) \, ds \right| \leq M |t - t_0| \leq Ma < b. \) (The last inequality follows from the choice of \( a \) in step (2).)

Facts a-c show \( T \) maps the space \( Y \) into itself.

(d) \( z'(t) = f(t, y(t)) \) (proof: fundamental theorem of calculus).

(e) Definition: the function \( y \in Y \) is a fixed point of \( T \) means \( Ty = y \).  

Claim: \( y \) is a solution to the IVP \( \iff \) \( y \) is a fixed point of \( T \).

Proof: Suppose \( y \) is a solution, i.e., \( y(t_0) = y_0 \) and \( y'(t) = f(t, y(t)) \). Then

\[
Ty = y_0 + \int_{t_0}^{t} f(s, y(s)) \, ds
\]

\[
= y_0 + \int_{t_0}^{t} y'(s) \, ds = y_0 + y(s)|_{t_0}^{t} = y(t)
\]

So \( y \) is a fixed point of \( T \).

Conversely, suppose \( y \) is a fixed point, then \( y = Ty = y_0 + \int_{t_0}^{t} f(s, y(s)) \, ds \).

\[\Rightarrow y(t_0) = y_0 \quad \text{and} \quad y' = f(t, y(t)). \] I.e. \( y \) satisfies the IVP. QED

The claim shows that proving existence and uniqueness is equivalent to proving that \( T \) has a unique fixed point. (This is proved in (8) and (9) below.)

(4) **The metric on \( Y \)**

For \( y_1 \) and \( y_2 \) in \( Y \) define

\[
\delta(y_1, y_2) = \max_{[t_0-a, t_0+a]} |y_1(t) - y_2(t)|.
\]

(a) \( \delta(y_1, y_2) = 0 \iff y_1 = y_2 \) \ ((proof: trivial).

(b) \( \delta \) satisfies the triangle inequality: \( \delta(y_1, y_2) + \delta(y_2, y_3) \geq \delta(y_1, y_3) \) \ (proof: not hard).

(c) \( \delta \) tells how to measure ’closeness’ between ’points’ of \( Y \).

(d) \( Y \) has no ’holes’.

Formally: \( Y \) is a complete metric space.

Informally: Complete means all Cauchy sequences converge.

Analogy: On the real line \( R \) the sequence \( 1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, \ldots \) converges to 1.

However, in \( R - \{1\} \) the sequence doesn’t converge. This is because \( R - \{1\} \) has a ’hole’ at 1.

For us it is enough to know that if the sequence \( y_0, y_1, \ldots \) satisfies \( \sum_{n=1}^{\infty} \delta(y_{n+1}, y_n) < \infty \) then the sequence converges, i.e., \( \lim_{n \to \infty} y_n = y \) exists.

(Picture: each step from \( y_0 \) to \( y_1 \) to \( y_2 \) etc gets smaller in such a way that the total distance is finite.)
Completeness is not hard to show. It does require a careful \( \epsilon - \delta \) proof.

(5) **Claim:** \( \delta(Ty_1, Ty_2) \leq \frac{1}{2}\delta(y_1, y_2) \).

**Proof:**

\[
|Ty_1(t) - Ty_2(t)| = \left| \int_{t_0}^{t} f(s, y_1(s)) - f(s, y_2(s)) \, ds \right|
\leq \int_{t_0}^{t} |f(s, y_1(s)) - f(s, y_2(s))| \, ds
\leq L \int_{t_0}^{t} |y_1(s) - y_2(s)| \, ds \quad \text{(Lipschitz condition)}
\leq L\delta(y_1, y_2) \int_{t_0}^{t} ds \quad \text{(pull out max\( (y_1(s) - y_2(s)) \))}
\leq L\delta(y_1, y_2)(t - t_0) \leq \delta(y_1, y_2) L \cdot a < \frac{1}{2}\delta(y_1, y_2) \quad \text{QED}
\]

The last inequality uses the choice of \( a \) in step (2).

Note: since \( T \) shrinks distances it is called a **contraction mapping**.

By way of analogy let \( A : [0, 1] \rightarrow [0, 1] \) by the formula \( A(x) = \frac{1}{2}(1 - x) \). It’s easy to see \( |A(x_1) - A(x_2)| = \frac{1}{2}|x_1 - x_2| \), so \( A \) is a contraction mapping. If you start with the whole interval \([0, 1]\) and repeatedly apply \( A \) you get a sequence of intervals \([0, 1], [0, 1/2], [1/4, 1/2], [1/4, 3/8]\). These intervals get smaller and smaller, shrinking down to the fixed point \( x = 1/3 \).

(6) **Claim:** \( T \) has at most one fixed point.

**Proof:** Suppose there were two different fixed points \( y_1 \) and \( y_2 \). Then since \( Ty_j = y_j \) we get \( \delta(Ty_1, Ty_2) = \delta(y_1, y_2) \). But, this contradicts (5) where we saw \( \delta(Ty_1, Ty_2) \leq \frac{1}{2}\delta(y_1, y_2) \).

(7) If the sequence \( y_0, y_1, y_2, \ldots \) converges to \( y \) then \( Ty_0, Ty_1, Ty_2, \ldots \) converges to \( Ty \).

Formally: \( T \) is a continuous map of \( Y \) to itself.

(8) **Picard iteration**

Start with \( y_0(t) = y_0 \). Let \( y_1 = Ty_0, y_2 = Ty_1, \ldots, y_{n+1} = Ty_n = T^n y_0 \).

**Claim:** the sequence \( y_0, y_1, \ldots \) converges.

**Proof:**

\[
\delta(y_2, y_1) = \delta(Ty_1, Ty_0) \leq \frac{1}{2}\delta(y_1, y_0)
\]

Likewise, \( \delta(y_3, y_2) = \delta(Ty_2, Ty_1) \leq \frac{1}{2}\delta(y_2, y_1) \leq \frac{1}{4}\delta(y_1, y_0) \)

Generally, \( \delta(y_{n+1}, y_n) \leq \left( \frac{1}{2} \right)^n \delta(y_1, y_0) \)

\[
\Rightarrow \sum_{n=0}^{\infty} \delta(y_{n+1}, y_n) \leq \delta(y_1, y_0) \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n .
\]

Since this last sum converges the completeness of \( Y \) proves the claim.

(9) Take the sequence from (8) and let \( \lim_{n \rightarrow \infty} y_n = y \).

**Claim:** \( y \) is a fixed point of \( T \).

**Proof:** Since \( y = \lim T^n y_0 \) we have \( Ty = \lim T^{n+1} y_0 = y \). QED
**Example:** (Picard iteration) Consider the IVP $y' = y, \quad y(0) = 1$.

Picard iteration gives $y_0(t) = 1$.

$$y_1(t) = y_0 + \int_0^t 1 \, ds = 1 + t.$$  

$$y_2(t) = y_0 + \int_0^t 1 + t \, ds = 1 + t + \frac{t^2}{2}.$$  

$\Rightarrow$ In this case Picard iteration leads to the power series for $e^t$. 