Gibbs’ Phenomenon says that the truncated Fourier series near a jump discontinuity overshoots the jump by about 9% of the size of the jump. Thus, for the standard square wave (which jumps between -1 and 1) the peak value of the truncated Fourier series is about 1.18.

The proof is an elaborate and tricky calculus exercise.

Start with the square wave
\[ f(x) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nx)}{n} \]

The truncated Fourier series is
\[ f_N(x) = \frac{4}{\pi} \left( \sin(x) + \frac{\sin(3x)}{3} + \ldots + \frac{\sin((2N-1)x)}{2N-1} \right) \]

Taking the derivative
\[ f'_N(x) = \frac{4}{\pi} \left( \cos(x) + \cos(3x) + \ldots + \cos((2N-1)x) \right) \]

**Claim:** \[ f'_N(x) = \frac{2}{\pi} \cdot \frac{\sin(2Nx)}{\sin(x)}. \]

**Proof:** Using complex arithmetic:
\[ \cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}. \]

\[ \Rightarrow f'_N(x) = \frac{4}{\pi} \left( \frac{e^{-2N+1}ix + e^{-2N-3}ix + \ldots + e^{(2N-3)ix} + e^{(2N-1)ix}}{2} \right). \]

This is a geometric series with ratio \( e^{2ix} \Rightarrow \)
\[ f'_N(x) = \frac{2}{\pi} \cdot \frac{e^{(-2N+1)ix} - e^{(2N+1)ix}}{1 - e^{2ix}}. \]

Multiply top and bottom by \( e^{-ix} \) and use the formula for \( \sin(x) \) in terms of complex exponentials to get
\[ f'_N(x) = \frac{2}{\pi} \cdot \frac{e^{-2N}ix - e^{(2N)ix}}{e^{-ix} - e^{ix}} = \frac{2}{\pi} \cdot \frac{\sin(2Nx)}{\sin(x)}. \quad \text{QED} \]

The maximum overshoot of \( f_N(x) \) is its first positive maximum. The formula for \( f'_N(x) \) shows this is at \( \frac{\pi}{2N} \). Since, \( f_N(0) = 0 \) and all the terms in the sum for \( f_N(\pi/2N) \) are positive we conclude that \( x = \pi/2N \) is a local maximum (it is, in fact, the absolute maximum).

We now set about estimating \( f_N(\pi/2N) \), i.e., the maximum value of \( f_N(x) \). First we manipulate the series for \( f_N(\pi/2N) \).

\[ f_N \left( \frac{\pi}{2N} \right) = \frac{4}{\pi} \left( \frac{\sin(\pi/2N)}{1} + \frac{\sin(3\pi/2N)}{3} + \ldots + \frac{\sin((2N-1)\pi/2N)}{2N-1} \right) \]
\[ = \frac{2}{\pi} \cdot \frac{\pi}{N} \left( \frac{\sin(\pi/2N)}{\pi/2N} + \frac{\sin(3\pi/2N)}{3\pi/2N} + \ldots + \frac{\sin((2N-1)\pi/2N)}{(2N-1)\pi/2N} \right) \]
This last is a Riemann sum (using midpoints) for
\[ \frac{2}{\pi} \int_0^\pi \frac{\sin(x)}{x} \, dx, \text{ with } \Delta x = \pi/N. \]

Since \( \Delta x \to 0 \) as \( N \to \infty \) we get
\[ \lim_{N \to \infty} f_N(\pi/2N) = \frac{2}{\pi} \int_0^\pi \frac{\sin(x)}{x} \, dx. \]

In words, as \( N \) increases the overshoot goes to the value of the integral.

All that’s left is to estimate the value of the integral. For this we integrate the power series for \( \frac{\sin(x)}{x} \). We have
\[ \frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots \]

Which gives
\[ \frac{2}{\pi} \int_0^\pi \frac{\sin(x)}{x} \, dx = 2 \left( 1 - \frac{\pi^2}{3 \cdot 3!} + \frac{\pi^4}{5 \cdot 5!} - \frac{\pi^6}{7 \cdot 7!} + \ldots \right) \]

This series converges very rapidly and after five terms we have the value 1.18 correct to 2 decimal places.

We have seen that as \( N \) gets large the maximum value of \( f_N(x) \) becomes 1.18. That is it overshoots the correct value by 0.18, which is 9% of the jump from -1 to 1.