**18.03 Gibbs’ Phenomenon**
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Gibbs’ Phenomenon says that the truncated Fourier series near a jump discontinuity overshoots the jump by about 9% of the size of the jump. Thus, for the standard square wave (which jumps between -1 and 1) the peak value of the truncated Fourier series is about 1.18.

The proof is an elaborate and tricky calculus exercise.

Start with the square wave

$$ f(x) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nx)}{n} $$

The truncated Fourier series is

$$ f_N(x) = \frac{4}{\pi} \left( \sin(x) + \frac{\sin(3x)}{3} + \ldots + \frac{\sin((2N-1)x)}{2N-1} \right) $$

Taking the derivative

$$ f'_N(x) = \frac{4}{\pi} \left( \cos(x) + \cos(3x) + \ldots + \cos((2N-1)x) \right) $$

Claim: $$ f'_N(x) = \frac{2}{\pi} \frac{\sin(2Nx)}{\sin(x)} $$

Proof: Using complex arithmetic:

$$ \cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i} $$

$$ \Rightarrow f'_N(x) = \frac{4}{\pi} \left( \frac{e^{(-2N+1)ix} + e^{(-2N+3)ix} + \ldots + e^{(2N-3)ix} + e^{(2N-1)ix}}{2} \right). $$

This is a geometric series with ratio $e^{2ix}$:

$$ f'_N(x) = \frac{2}{\pi} \frac{e^{(-2N+1)ix} - e^{(2N+1)ix}}{1 - e^{2ix}}. $$

Multiply top and bottom by $e^{-ix}$ and use the formula for $\sin(x)$ in terms of complex exponentials to get

$$ f'_N(x) = \frac{2}{\pi} \frac{e^{(-2N)ix} - e^{(2N)ix}}{e^{-ix} - e^{ix}} = \frac{2}{\pi} \frac{\sin(2Nx)}{\sin(x)}, \quad \text{QED} $$

The maximum overshoot of $f_N(x)$ is its first positive maximum. The formula for $f'_N(x)$ shows this is at $\frac{\pi}{2N}$. Since, $f_N(0) = 0$ and all the terms in the sum for $f_N(\pi/2N)$ are positive we conclude that $x = \pi/2N$ is a local maximum (it is, in fact, the absolute maximum).

We now set about estimating $f_N(\pi/2N)$, i.e., the maximum value of $f_N(x)$. First we manipulate the series for $f_N(\pi/2N)$.

$$ f_N \left( \frac{\pi}{2N} \right) = \frac{4}{\pi} \left( \frac{\sin(\pi/2N)}{1} + \frac{\sin(3\pi/2N)}{3} + \ldots + \frac{\sin((2N-1)\pi/2N)}{2N-1} \right) $$

$$ = \frac{2}{\pi} \frac{\pi}{N} \left( \frac{\sin(\pi/2N)}{\pi/2N} + \frac{\sin(3\pi/2N)}{3\pi/2N} + \ldots + \frac{\sin((2N-1)\pi/2N)}{(2N-1)\pi/2N} \right) $$

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This last is a Riemman sum (using midpoints) for
\[
\frac{2}{\pi} \int_{0}^{\pi} \frac{\sin(x)}{x} \, dx, \text{ with } \Delta x = \pi/N.
\]
Since $\Delta x \to 0$ as $N \to \infty$ we get
\[
\lim_{N \to \infty} f_N(\pi/2N) = \frac{2}{\pi} \int_{0}^{\pi} \frac{\sin(x)}{x} \, dx.
\]
In words, as $N$ increases the overshoot goes to the value of the integral.

All that’s left is to estimate the value of the integral. For this we integrate the power series
for $\frac{\sin(x)}{x}$. We have
\[
\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots
\]
Which gives
\[
\frac{2}{\pi} \int_{0}^{\pi} \frac{\sin(x)}{x} \, dx = 2 \left(1 - \frac{\pi^2}{3 \cdot 3!} + \frac{\pi^4}{5 \cdot 5!} - \frac{\pi^6}{7 \cdot 7!} + \ldots\right)
\]
This series converges very rapidly and after five terms we have the value 1.18 correct to 2 decimal places.

We have seen that as $N$ gets large the maximum value of $f_N(x)$ becomes 1.18. That is it overshots the correct value by 0.18, which is 9% of the jump from -1 to 1.