18.03 The Heat Equation
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1 Introduction

The goal in this note is to derive the heat equation from Newton’s law of cooling. We can get most of the way using just mathematics. The final piece of the puzzle requires the use of an empirical physical principle of heat flow.

In outline: First we’ll set up the problem of heat flow in a bar. Then will discretize the problem and analyze $n \times n$ systems of equations based on Newton’s law of cooling. Finally, we’ll let get the continuous heat equation as the limit of the discretized system as the discrete stepsize goes to zero.

2 A heated bar

Suppose we have a heated bar where the temperature varies with position along the bar as well as over time. To be specific, we assume we have a rod of length $L$ which is thin enough that the temperature doesn’t vary in the vertical direction. So we can describe the temperature of the bar by a function of two variables $u(t, x)$ which gives the temperature at time $t$ at position $x$.

For this example we will also assume that the sides of the bar are insulated so that no heat passes through them. We will also assume that the ends of the bar are kept in an ice bath at $0^\circ$. This is illustrated in the figure below.

Heated rod with temperature varying by position and time: $u(t, x)$.

As in previous examples in the course, let’s divide the rod into sections and assume the temperature is uniform in each section.

Heated rod divided into pieces.

Let’s call the temperature in the $j$th section $u_j(t)$. Newton’s law of cooling gives

$$u_j' = -k(u_j - u_{j-1}) - k(u_j - u_{j+1}) = ku_{j-1} - 2ku_j + ku_{j+1}$$
Taking into account the ice baths we get a system of differential equations. (We show the coefficient matrix for $n = 4$. It follows the same pattern for other values of $n$.)

$$u' = Au,$$

where

$$u = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix}, \quad \text{and (for } n = 4) \quad A = \begin{bmatrix} -2k & k & 0 & 0 \\ k & -2k & k & 0 \\ 0 & k & -2k & k \\ 0 & 0 & k & -2k \end{bmatrix}$$

We’ll call this equation the discrete heat equation because we’ve divided the bar into discrete chunks.

For future reference note the way we divided the bar into sections: The $n$ middle pieces all have width $\Delta x = L/(n+1)$ and centers $x_j = j\Delta x$ and the two end pieces (in the ice baths) are each half that width.

3 Eigenvalues and eigenvectors of the matrix $A$

We know that to solve the discrete heat equation we must find eigenvalues and eigenvectors of the coefficient matrix $A$. These are given by

Theorem: Let

$$\theta_m = \frac{m\pi}{n+1} \quad \text{and} \quad \lambda_m = -2k + 2k \cos(\theta_m)$$

Then $A$ has eigenvalues $\lambda_m$ for $m = 1, 2, \ldots, n$ and corresponding eigenvectors

$$v_m = \begin{bmatrix} \sin(\theta_m) \\ \sin(2\theta_m) \\ \vdots \\ \sin(n\theta_m) \end{bmatrix}$$

Proof number 1: This can be checked directly using the trig identity

$$\sin((k-1)\theta) + \sin((k+1)\theta) = 2\cos(\theta)\sin(k\theta)$$

The exact value of $\theta_m$ only comes into play for the $n$th entry in the eigenvector. For that entry the eigenequation takes the form

$$\sin((n-1)\theta_m) - 2\sin(n\theta_m) = (-2 + 2\cos(\theta_m))\sin(n\theta_m)$$

This follows from the previous trig identity with $k = n$ because for this particular $\theta_m$ we have $\sin((n+1)\theta_m) = 0$.

This proof doesn’t give much insight into how we might discover these eigenthings. The entire next section will be devoted to that.
4 Derivation via a boundary value problem

The presentation will be a little cleaner if we remove the diagonal from $A$. We write $A = -2kI + kB$. For example when $n = 4$:

\[
B = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}, \quad \text{so} \quad A = \begin{bmatrix}
-2k & k & 0 & 0 \\
k & -2k & k & 0 \\
0 & k & -2k & k \\
0 & 0 & k & -2k
\end{bmatrix} = -2k \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} + k \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

It is easy to see that if $v$ is an eigenvector of $B$ with eigenvalue $\lambda$ then the eigenequation $Bv = \lambda v$ implies

\[
Av = (-2kI + kB)v = (-2k + k\lambda)v.
\]

That is, $v$ is an eigenvector of $A$ with eigenvalue $-2k + k\lambda$.

With this in mind we set about deriving the eigenvalues and eigenvectors of $B$. Write

\[
v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \quad \text{The eigenequation } Bv = \lambda v \text{ can be written out as}
\]

\[
\begin{align*}
 v_2 &= \lambda v_1 \\
v_1 + v_3 &= \lambda v_2 \\
v_2 + v_4 &= \lambda v_3 \\
&\vdots \\
v_{n-2} + v_n &= \lambda v_{n-1} \\
v_{n-1} &= \lambda v_n
\end{align*}
\]

Except for the first and last equations all these equations look the same. We can make all the equations look the same by introducing $v_0$ and $v_{n+1}$:

\[
\begin{align*}
v_0 + v_2 &= \lambda v_1 \\
v_1 + v_3 &= \lambda v_2 \\
v_2 + v_4 &= \lambda v_3 \\
&\vdots \\
v_{n-2} + v_n &= \lambda v_{n-1} \\
v_{n-1} + v_{n+1} &= \lambda v_n
\end{align*}
\]

To make sure this doesn’t change anything we must require that

\[
v_0 = 0 \quad \text{and} \quad v_{n+1} = 0. \quad (BC)
\]

The eigenequations above have the form

\[
v_{j-1} + v_{j+1} = \lambda v_j \quad \text{for} \quad j = 1, 2, \ldots n
\]

We rewrite them as

\[
v_{j-1} - \lambda v_j + v_{j+1} = 0 \quad \text{for} \quad j = 1, 2, \ldots n \quad (\Delta \text{eq.})
\]
This is called a difference equation for the $v_j$.

The conditions (BC) are called boundary conditions. The name boundary conditions indicates that they are on the boundary, that is ends, of the vector. Physically $v_0$ and $v_{n+1}$ correspond to the pieces in ice baths at the end of the rod in positions $x_0$ and $x_{n+1}$ respectively.

**To summarize:** we’ve recast the eigenequation as a difference equation with boundary conditions.

### 4.1 Solving the difference equation

Our first goal will be to solve the difference equation (∆ eq.) without regard for the boundary conditions. We use our usual method of optimism to find modal solutions—in the case of difference equations these are of the form $v_j = a^j$. Substituting this in the equation (∆ eq.) we get

$$a^{j-1} - \lambda a^j + a^{j+1} = 0.$$

Factoring out $a^{j-1}$ we get the characteristic equation

$$1 - \lambda a + a^2 = 0.$$

The quadratic formula now yields $a = \frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}$ By reversing the order under the square root we can write this in the form

$$a = \frac{\lambda}{2} \pm i \frac{\sqrt{4 - \lambda^2}}{2}.$$

Now comes a clever trick: we notice that $\left(\frac{\lambda}{2}\right)^2 + \left(\frac{\sqrt{4 - \lambda^2}}{2}\right)^2 = 1$. So, for $\lambda \leq 2$ we can choose $\theta$ so that

$$\frac{\lambda}{2} = \cos(\theta), \quad \frac{\sqrt{4 - \lambda^2}}{2} = \sin(\theta), \quad \text{and} \quad a = \cos(\theta) \pm i \sin(\theta) = e^{\pm i \theta}. \quad (1)$$

The general solution to the difference equation is a superposition of the two modal solutions

$$v_j = c_1 e^{ij \theta} + c_2 e^{-ij \theta} \quad \text{valid for} \quad j = \ldots -2, -1, 0, 1, 2, \ldots$$

(We won’t go into it, but since the characteristic equation is second order the space of solutions to the difference equation is two dimensional.)

### 4.2 Satisfying the boundary conditions

Notice that we solved the difference equation (∆ eq.) for every value of $\lambda \leq 2$. (It will turn out that this is good enough since we will find enough eigenvalues of this form.) Now we have to figure out for which of these $\lambda$ the solutions can also satisfy the boundary conditions (BC). Since such $\lambda$ are the eigenvalues of $B$ we know there are only $n$ of them.
Substituting the general solution of \((\Delta \text{ eq.})\) into the boundary conditions we get

\[ v_0 = c_1 + c_2 = 0 \]
\[ v_{n+1} = c_1 e^{i(n+1)\theta} + c_2 e^{-i(n+1)\theta} = 0 \]

Solving we get

\[ c_1 = -c_2 \quad \text{and} \quad e^{i(n+1)\theta} - e^{-i(n+1)\theta} = 0 \]

The difference of exponentials \(e^{i(n+1)\theta} - e^{-i(n+1)\theta} = 2i \sin((n+1)\theta)\). This is 0 exactly when \((n + 1)\theta = m\pi\) for some integer \(m\). We want \(n\) distinct eigenvalues which we get from Equation 1:

\[ \lambda_m = 2 \cos(\theta_m), \quad \text{with} \quad \theta_m = \frac{m\pi}{n+1} \quad \text{for} \quad m = 1, 2, \ldots, n. \]

Converting from eigenvalues of \(B\) to eigenvalues of \(A\) we see that the \(n\) eigenvalues of \(A\) are

\[ -2k + k\lambda_m = -2k + 2k \cos(\theta_m) \quad \text{for} \quad m = 1, 2, \ldots, n. \]

Using the relation \(c_1 = -c_2\) found above, the corresponding eigenvectors are

\[ c_1 \begin{bmatrix} e^{i\theta_m} \\ e^{i2\theta_m} \\ \vdots \\ e^{in\theta_m} \end{bmatrix} - c_1 \begin{bmatrix} e^{-i\theta_m} \\ e^{-i2\theta_m} \\ \vdots \\ e^{-in\theta_m} \end{bmatrix} = c_1 \begin{bmatrix} e^{i\theta_m} - e^{-i\theta_m} \\ e^{i2\theta_m} - e^{-i2\theta_m} \\ \vdots \\ e^{in\theta_m} - e^{-in\theta_m} \end{bmatrix} = 2ic_1 \begin{bmatrix} \sin(\theta_m) \\ \sin(2\theta_m) \\ \vdots \\ \sin(n\theta_m) \end{bmatrix} \]

Removing the factor of \(2ic_1\) these are exactly the eigenvectors claimed in the theorem.

We have found all the eigenthings needed to give the general solution to the discrete heat equation \(u' = Au\). We record it here:

\[ u = c_1 e^{\lambda_1 t} v_1 + \ldots + c_n e^{\lambda_n t} v_n, \quad \text{where} \quad \theta_m = \frac{m\pi}{n+1}, \lambda_m = -2 + 2 \cos(\theta_m), \quad v_m = \begin{bmatrix} \sin(\theta_m) \\ \sin(2\theta_m) \\ \vdots \\ \sin(n\theta_m) \end{bmatrix} \]

5 Letting \(n\) go to infinity

The model that a given section of the heated rod has a uniform temperature is clearly flawed. We can make it more accurate by dividing the rod into more and more pieces. In the limit the pieces become infinitesimal and the model becomes exact. (Well exact assuming the linear assumptions in Newton’s law of cooling are true.)

While we proceed you should remember that the entries in the vector \(u(t)\) represent the temperature at \(x\) positions along the rod. In the limit the vector will become the heat function \(u(t,x)\) giving the temperature at every point along the rod.
5.1 The rate constant $k$

Physically it makes more sense to talk about the movement of heat content between the sections than the movement of temperature. Since heat content is the integral of temperature we were able to write out equations in terms of temperature. But this makes the rate constant $k$ dependent on the dimensions of the sections as well as the physical properties of the material in the rod.

We know that as $\Delta x$ decreases there is less distance for the heat to move so $k$ should increase. Empirically it turns out that

$$k = \frac{k_0}{(\Delta x)^2}$$

where $k_0$ is a physical constant associated to the material but not the dimensions of the sections. (See the appendix for a derivation of this from another physical principle.)

5.2 The limit of the modal solutions to the discrete heat equation

The computations to do this limiting procedure are a little involved, but they follow the standard methods from calculus. We have

$$L = \text{length of rod}$$

$$\Delta x = \frac{L}{n+1} = \text{width of the segments}$$

$$x_j = j\Delta x = \frac{jL}{n+1} = \text{position of the } j\text{th segment}$$

$$\sin(j\theta_m) = \sin\left(j \cdot \frac{m\pi}{n+1}\right) = \sin\left(\frac{m\pi}{L} \cdot x_j\right)$$

As $n$ increases the points $x_j = j\Delta x$ fill in the rod. So the vector

$$v_m = \begin{bmatrix} \sin(\theta_m) \\ \sin(2\theta_m) \\ \vdots \\ \sin(n\theta_m) \end{bmatrix} = \begin{bmatrix} \sin(m\pi x_1/L) \\ \sin(m\pi x_2/L) \\ \vdots \\ \sin(m\pi x_n/L) \end{bmatrix}$$

can be replaced by the function $\sin(m\pi x/L)$.

Likewise for the eigenvalue $\lambda_m$ we have

$$\lambda_m = \left(-2 + 2\cos\left(\frac{m\pi}{n+1}\right)\right) k$$

$$= \left(-2 + 2\cos\left(\frac{m\pi}{n+1}\right)\right) \frac{k_0}{(\Delta x)^2}$$

$$= \left(-2 + 2\cos\left(\frac{m\pi}{n+1}\right)\right) \frac{k_0(n+1)^2}{L^2}$$
Using the power series for \( \cos \left( \frac{m\pi}{n+1} \right) \)

\[
= \left( \frac{2}{2} - \frac{2 \left( \frac{m\pi}{n+1} \right)^2}{4!} - \cdots \right) \frac{k_0(n+1)^2}{L^2}
\]

\[= - \left( \frac{m\pi}{L} \right)^2 k_0 + \text{terms with powers of } \frac{1}{(n+1)^2} \]

From the last line we see that in the limit as \( n \) goes to infinity \( \lambda_m \) goes to \( - \left( \frac{m\pi}{L} \right)^2 k_0 \).

Putting together the limits of \( v_m \) and \( \lambda_m \) we see that the modal solution to the discrete heat equation

\[ u = e^{-\lambda_m t} v_m \]

goes to the continuous heat function as \( n \) goes to infinity

\[ u(t, x) = e^{-(m\pi/L)^2 k_0 t} \sin \left( \frac{m\pi}{L} x \right). \] (2)

6 The continuous heat equation

The discrete heat function goes in the limit to the continuous heat function. We will now show that the discrete heat equation limits to a differential equation called the continuous heat equation.

In the discrete heat equation we have

\[ u'_j = k(u_{j-1} - 2u_j + u_{j+1}) \quad \text{for} \quad j = 1, 2, \ldots, n \]

where just as we did above we use \( u_0(t) \) and \( u_{n+1}(t) \) with the boundary conditions

\[ u_0(t) = 0 \quad \text{and} \quad u_{n+1}(t) = 0. \]

Substituting for \( k \) gives

\[ u'_j = \frac{k_0}{(\Delta x)^2} (u_{j-1} - 2u_j + u_{j+1}) = \frac{k_0}{(\Delta x)^2} \frac{u_{j-1} - 2u_j + u_{j+1}}{(\Delta x)^2} \] (3)

Since \( u_j(t) \) is the temperature of the rod at \( x_j = j\Delta x \) we can write \( u_j(t) = u(t, j\Delta x) \). Therefore the expression \( u_{j-1} - 2u_j + u_{j+1} \) is a second difference in the \( x \) variable, that is

\[ u_{j-1} - 2u_j + u_{j+1} = u(t, (j-1)\Delta x) - 2u(t, j\Delta x) + u(t, (j+1)\Delta x) \approx \frac{\partial^2 u}{\partial x^2}(t, j\Delta x) \cdot \Delta x^2 \] (4)

Likewise

\[ u'_j(t) = \frac{\partial u}{\partial t}(t, j\Delta x). \] (5)

Using (5) and (4) equation (3) becomes

\[ \frac{\partial u}{\partial t}(t, j\Delta x) \approx k_0 \frac{\partial^2 u}{\partial x^2}(t, j\Delta x) \]
Now, in the limit as \( n \) goes to infinity this equation becomes exact
\[
\frac{\partial u}{\partial t}(t, x) = k_0 \frac{\partial^2 u}{\partial x^2}(t, x).
\]
This is a partial differential equation (PDE). It is called the continuous heat equation—or simply the heat equation. Finally in the limit the boundary conditions become
\[
u(t, 0) = 0 \quad \text{and} \quad u(t, L) = 0. \quad \text{(Continuous BC)}
\]

### 6.1 General solution of the continuous heat equation

We saw in equation \ref{eq:modal_solution} the limit of the modal solution \( u = e^{\lambda_m t}v_m \) to the discrete heat equation with boundary conditions (BC). This limit must be a solution to the continuous heat equation with boundary conditions (Continuous BC). Taking the limit of the general solution to the discrete heat equation with boundary conditions (BC) gives us the general solution to the continuous heat equation with boundary conditions (Continuous BC):
\[
u(x, t) = \sum_{m=1}^{\infty} c_m e^{-\frac{(m\pi/L)^2 K t}{2}} \sin\left(\frac{m\pi}{L} x\right).
\]

### 7 Appendix

#### 7.1 The empirical principle

We made use of the empirical principle \( k = \frac{k_0}{(\Delta x)^2} \), where \( k \) is the rate constant in the discrete heat equation and \( k_0 \) is a physical constant associated with the material of the rod but not its dimensions.

Here we will show this is equivalent to the empirical principle used in Edwards and Penney section 8.5 to derive the continuous heat equation directly. Their principle is
\[
\phi(t, x) = -K \frac{\partial u}{\partial x}(t, x)
\]
where \( \phi \) is the heat flux (heat per area per time) through the cross-section at position \( x \) and time \( t \) and \( K \) is the thermal conductivity of the material. (Flux is positive from right to left.)

To connect their principle and ours we need to know that the heat content in a section of rod from \( a \) to \( b \) is
\[
Q = \int_a^b c\delta S u(t, x) \, dx
\]
where \( c \) is the specific heat of the material, \( \delta \) the density of the rod and \( S \) the cross-sectional area. If \( \Delta x = b - a \) is small then
\[
Q \approx c\delta S u(t, a) \Delta x.
\]
The net heat flux into the section of rod is therefore
\[
\frac{1}{S} \frac{\partial Q}{\partial t} \approx c\delta \frac{\partial u}{\partial t}(t, a) \Delta x
\]
The net heat flux into the section is the difference between the fluxes at either end. So, using the principle from Edwards and Penney (we have to be careful with signs):

\[
\text{net flux in} = K \left( \frac{\partial u}{\partial x}(t,b) - \frac{\partial u}{\partial x}(t,a) \right) \approx K \frac{\partial^2 u}{\partial x^2}(t,a) \Delta x.
\]

Equating these two formulas for net flux we get

\[
c^\delta \frac{\partial u}{\partial t}(t,a) \Delta x \approx K \frac{\partial^2 u}{\partial x^2}(t,a) \Delta x. \tag{6}
\]

Now we approximate the second partial derivative by a second difference

\[
\frac{\partial^2 u}{\partial x^2}(t,a) \approx \frac{u(t,a - \Delta x) - 2u(t,a) + u(t,a + \Delta x)}{\Delta x^2}
\]

Equation 6 becomes

\[
\frac{\partial u}{\partial t}(t,a) \approx \frac{K}{c^\delta} \frac{\partial^2 u}{\partial x^2}(t,a)
\]

\[
\approx \frac{K}{c^\delta} \frac{u(t,a - \Delta x) - 2u(t,a) + u(t,a + \Delta x)}{\Delta x^2}
\]

\[
= \frac{K}{c^\delta(\Delta x)^2} (u(t,a - \Delta x) - 2u(t,a) + u(t,a + \Delta x))
\]

Now if \( a = x_j, \ a - \Delta x = x_{j-1} \) and \( a + \Delta x = x_{j+1} \) then this becomes our discrete heat equation

\[
\frac{\partial u}{\partial t}(t,x_j) \approx \frac{K}{c^\delta(\Delta x)^2} (u(t,x_{j-1}) - 2u(t,x_j) + u(t,x_{j+1}))
\]

Comparing constants we see that \( k = \frac{K}{c^\delta(\Delta x)^2} \), so letting \( k_0 = \frac{K}{c^\delta} \) we get

\[
k = \frac{k_0}{\Delta x^2}
\]

which is exactly the principle we asserted.