17 Matrix methods for solving systems of DEs

17.1 Goals

1. Be able to solve constant coefficient linear systems using eigenvalues and eigenvectors. Do this with when there are real or complex eigenvalues.

2. Understand and appreciate the abstraction of matrix notation.

3. Be able to convert a higher order linear DE equation into a companion system of coupled first order equations.

4. See some physical settings modeled by systems of equations.

17.2 Introduction

In this topic we will look in detail at solving linear constant coefficient systems of differential equations using eigenvalues and eigenvectors. We will need to consider cases of real, complex and repeated eigenvalues. (We will really only touch on the case of repeated eigenvalues.)

An important idea is that any higher order differential equation can be converted into a system of first order equations. This means that our old friend $P(D)x = 0$ can be converted into a system and solved with these methods. This is useful because writing code to solve first order systems is more natural than code for higher order equations. This is partly explained by the first section which looks at the utility of matrix notation.

17.3 Matrix notation and why we like it

We have been using matrix notation for algebraic systems and systems of differential equations. Let’s remind ourselves why it’s helpful in organizing our thinking.

One of the simplest algebraic equations is

$$ax = b,$$

where $a$ and $b$ are constants and $x$ is the unknown. \hspace{1cm} (1)

We easily solve this for $x$: $x = a^{-1}b$ (provided $a \neq 0$).

On the face of it a system of algebraic equations seem more complicated. For example consider the following system of two equations in two unknowns:

$$6x + 5y = 2$$
$$x + 2y = 3$$

We could solve this by elimination, but here our interest in writing this out abstractly. In matrix form the system and its solution become

$$\begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

1
If we give names: \( A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}, \ x = \begin{bmatrix} x \\ y \end{bmatrix}, \ b = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \) then the system and its solution become

\[ A x = b \Rightarrow x = A^{-1} b. \]

At this level of abstraction we see that the system and its solution are just like those of our simplest equation.

(One small difference is that we need to take more care with the order of matrix multiplication than we do with scalar multiplication.)

For differential equations our simplest and favorite equation is

\[ x' = a x. \]

Written in matrix form a linear system of DEs looks similar.

**Example 17.1.** As above, let \( A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \) and \( x = \begin{bmatrix} x \\ y \end{bmatrix} \). Write the following system in a form that resembles our favorite DE.

\[ \begin{align*}
    x' &= 6x + 5y \\
y' &= x + 2y
\end{align*} \]

**answer:** In matrix form this becomes

\[ \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{or} \quad \mathbf{x}' = A \mathbf{x}. \]

The right hand equation looks just like our favorite DE.

Note: we will call \( A \) the **coefficient matrix** of the system.

### 17.4 Solving homogeneous DEs using matrix methods

#### 17.4.1 Review

In the previous topic we looked briefly at solving linear, homogeneous, constant coefficient systems using matrix methods. Recall that we used the method of optimism to guess a solution of the form \( e^{\lambda t} \mathbf{v} \). Substituting this in the equation leads immediately to the fact that \( \lambda \) must be an eigenvalue and \( \mathbf{v} \) an eigenvector.

We'll review the process with brief explanations. Later we will write model solutions that skip directly to the characteristic equation.

**Example 17.2.** Solve \( \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \).

This is a **linear, homogeneous, constant coefficient** system of DEs.

**answer:** Try \( \begin{bmatrix} x \\ y \end{bmatrix} = e^{\lambda t} \mathbf{v} \).
Substitution gives: $\lambda e^{\lambda t}v = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} e^{\lambda t}v \iff \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} v = \lambda v.$

The boxed equation is the eigenvector/eigenvalue equation.

$\lambda$ is the eigenvalue and $v$ is the corresponding eigenvector.

We know how to find eigenvalues and eigenvectors.

**Characteristic equation:**

$$\begin{vmatrix} 3 - \lambda & 2 \\ 1 & 2 - \lambda \end{vmatrix} = 0 \iff \lambda^2 - 5\lambda + 4 = 0 \Rightarrow \lambda = 4, 1.$$

**Eigenvectors** $(A - \lambda I)v = 0$.

$\lambda_1 = 4$: $\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} v = 0$. Take $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$\lambda_2 = 1$: $\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} v = 0$. Take $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Two *modal* solutions are $x_1 = e^{4t}v_1 = e^{4t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $x_2 = e^{t}v_2 = e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

The general solution is $x = c_1 x_1 + c_2 x_2 = c_1 e^{4t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

**Example 17.3.**

**Note:** Each of the solutions $x = e^{\lambda t}v$ is called a normal mode or modal solution.

### 17.4.2 Complex eigenvalues

We handle complex eigenvalues in exactly the same manner as we did complex characteristic roots for ordinary differential equations.

**Theorem:** Suppose $A$ is a real matrix. Consider the DE: $x' = Ax$.

If $z$ is a complex solution to this DE then both the real and imaginary parts of $z$ are also solutions.

**Proof:** Suppose $z = x_1 + ix_2$ then

$$z' = Az \Rightarrow (x_1 + ix_2)' = A(x_1 + iA) \Rightarrow x_1' + ix_2' = Ax_1 + iAx_2.$$

If two complex numbers are equal then their real parts must be equal and so must the imaginary parts. Therefore the equation above shows

$$x_1' = Ax_1 \quad \text{and} \quad x_2' = Ax_2.$$

That is, $x_1$ and $x_2$ are solutions to the DE.

**Notes:** 1. The proof is just linearity written out the long way.

2. To be perfectly careful we should say that $x_1$ and $x_2$ are the real and imaginary parts of $z$, but this is clear from the context.

The next example illustrates the use of this theorem.

**Example 17.4.** Find the general real-valued solution to

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$
**answer:** First find eigenvalues $\lambda$ using the characteristic equation.

$$ |A - \lambda I| = \begin{vmatrix} 3 - \lambda & -5 \\ 2 & 1 - \lambda \end{vmatrix} = 0 $$

This gives us $\lambda^2 - 4\lambda + 13 = 0$. Solving we get $\lambda = 2 \pm 3i$. (Complex roots always come in conjugate pairs.)

Next we find eigenvectors $v$, i.e. we solve

$$(A - \lambda I)v = 0. \quad (2)$$

To do the computation we’ll let $v = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$.

$\lambda = 2 + 3i$: Equation 2 becomes

$$ \begin{bmatrix} 1 - 3i & -5 \\ 2 & -1 - 3i \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. $$

By inspection we can take $v_1 = \begin{bmatrix} 5 \\ 1 - 3i \end{bmatrix}$ for our solution.

Note: There is no need to compute the second eigenvector since it is just the complex conjugate of the first one.

This gives us a complex-valued solution

$$ z_1 = e^{(2+3i)t} \begin{bmatrix} 5 \\ 1 - 3i \end{bmatrix} = e^{2t} (\cos 3t + i \sin 3t) \begin{bmatrix} 5 \\ 1 - 3i \end{bmatrix} 
= e^{2t} \begin{bmatrix} 5 \cos 3t + i5 \sin 3t \\ \cos 3t + 3 \sin 3t + i(-3 \cos 3t + \sin 3t) \end{bmatrix} $$

Just for completeness we give its complex conjugate which is also a solution

$$ z_2 = \overline{z_1} = e^{(2-3i)t} \begin{bmatrix} 5 \\ 1 + 3i \end{bmatrix} = e^{2t} \begin{bmatrix} 5 \cos 3t - i5 \sin 3t \\ \cos 3t + 3 \sin 3t - i(-3 \cos 3t + \sin 3t) \end{bmatrix} $$

The theorem above tells us that The real and imaginary parts of $z_1$ are both solutions:

$$ x_1 = e^t \begin{bmatrix} 5 \cos 3t \\ \cos 3t + 3 \sin 3t \end{bmatrix} 
= e^t \begin{bmatrix} 5 \cos 3t \\ -3 \cos 3t + \sin 3t \end{bmatrix} $$

As always, the general real-valued solution is given by superposition

$$ x = c_1 x_1 + c_2 x_2 = c_1 e^t \begin{bmatrix} 5 \cos 3t \\ \cos 3t + 3 \sin 3t \end{bmatrix} + c_2 e^t \begin{bmatrix} 5 \sin 3t \\ -3 \cos 3t + \sin 3t \end{bmatrix}. $$
17.4.3 Repeated roots (2×2 case only)

Repeated eigenvalues, i.e. characteristic roots, complicate matters somewhat. We will study this by looking at two examples.

Example 17.5. (Complete case) Solve \[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} =
\begin{bmatrix}
5 & 0 \\
0 & 5
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]

answer: This is a diagonal matrix so the eigenvalues are \( \lambda = 5, 5 \).

For \( \lambda = 5 \) the matrix \( A - \lambda I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \). The nullspace of this matrix is all of \( \mathbb{R}^2 \). That is, every vector is an eigenvector i.e. the eigenspace is 2 dimensional. We only need to choose two independent eigenvectors. Can choose the standard basis vectors:

\[
v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

(any other independent pair would work as well).

Thus the general solution to the DE is \( x = c_1 e^{5t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^{5t} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \).

This is called the complete case because we have a full complement of basic solutions. That is, we have two independent solutions to our second order system.

The next example looks at the so-called defective case. The name comes from the following ideas. If a matrix has a repeated eigenvalue we would like an independent eigenvector for each time the eigenvalue is repeated. The matrix is defective if this is not the case.

Example 17.6. (Defective case) Solve \[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} =
\begin{bmatrix}
7 & -1 \\
4 & 3
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]

answer: First we find the eigenvalues, \( \lambda \). The characteristic equation is

\[
|A - \lambda I| = \lambda^2 - 10\lambda + 25 = 0.
\]

So the eigenvalues are repeated: \( \lambda = 5, 5 \).

Next we find the eigenvectors \( v \). As usual we need to solve \( (A - \lambda I)v = 0 \).

For \( \lambda = 5 \): this equation becomes

\[
\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

The row reduced echelon form (RREF) of the coefficient matrix is

\[
R = \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}
\]

This has only one free variable, so the eigenspace is only one dimensional. A basis is given by \( v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \).

This eigenvector gives us one solution to the DE \( x_1 = e^{5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \).
As we said, this case is defective. The system is second-order but the eigenmethods only found one solution. We’ll use a magic algorithm to find a second solution. Below we’ll see why the magic worked. You will need to take 18.06 (or even better 18.701) for some insight into how this was discovered.

The first step of the algorithm is to solve \((A - \lambda I)v_2 = v_1\). That is,

\[
\begin{bmatrix}
2 & -1 \\
4 & -2
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2
\end{bmatrix} = 
\begin{bmatrix}
1 \\
2
\end{bmatrix}
\]

Using row reduction (or by inspection) we find that one solution is

\[v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \].

The algorithm now tells us that a second solution to the DE is

\[x_2 = te^{5t}v_1 + e^{5t}v_2 = te^{5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} .\]

Now that we have two solutions we can give the general solution to the DE:

\[x = c_1x_1 + c_2x_2 \]

\[= c_1 e^{5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \left( te^{5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \]

**Abstract version of defective case**

The example above is complicated by actual computations. Here is the abstract version of the algorithm for the defective case. We check that the result is a solution by plugging it into the DE.

The algorithm uses two vectors

1. An eigenvector \(v_1\), i.e. \(Av_1 = \lambda v_1\)
2. A vector \(v_2\) that satisfies \(Av_2 = v_1\). (\(v_2\) is sometimes called a generalized eigenvector.)

Then we assert that \(x_1 = e^{\lambda t}v_1\) and \(x_2 = te^{\lambda t}v_1 + e^{\lambda t}v_2\) are independent solutions to the DE.

**Proof:** \(x_1\) is a the eigenvector solution. To check \(x_2\) is a solution we plug it into the DE and check that both sides of the equation are the same.

(left side) \(x_2' = te^{\lambda t}v_1 + e^{\lambda t}v_1 + \lambda e^{\lambda t}v_2 = \lambda te^{\lambda t}v_1 + e^{\lambda t}(v_1 + \lambda v_2)\)

(right side) \(Ax_2 = te^{\lambda t}Av_1 + e^{\lambda t}Av_2 = \lambda te^{\lambda t}v_1 + e^{\lambda t}(v_1 + \lambda v_2)\)

Comparing both sides we see that \(x_2' = Ax_2\). That is \(x_2\) is a solution.

### 17.5 Companion systems

Early in 18.03 we learned how to solve ordinary differential equations \(P(D)x = 0\). For example \(x'' + 8x' + 7x = 0\). In this section we will convert a higher order ordinary differential equation to a system of first order equations.

**Example 17.7.** Convert the ODE \(x'' + 8x' + 7x = 0\) to a system of first order equations.
**answer:** Introduce a second variable \( y = x' \). Our ODE then becomes

\[
y' + 8y + 7x = 0.
\]

Writing out the equations for \( x' \) and \( y' \) we get

\[
\begin{align*}
x' &= y, \\
y' &= -7x - 8y \quad \Leftrightarrow \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -7 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\end{align*}
\]

The system is called the **companion system** to the original ODE. We call the coefficient matrix the **companion matrix**.

We will sometimes refer to the method of converting an ODE to a system as **anti-elimination**. This is because elimination is a process of removing variables and equations, so anti-elimination is a process of adding variables and equations.

**Example 17.8.** Find the companion system for the ODE \( x''' + 2x'' + 5x' + 7x = 0 \).

**answer:** Let \( y = x' \) and \( z = y' = x'' \). The ODE becomes

\[
z' + 2z + 5y + 7x = 0.
\]

So our companion system is

\[
\begin{align*}
x' &= y, \\
y' &= z, \\
z' &= -7x - 5y - 2z \quad \Leftrightarrow \quad \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -5 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\end{align*}
\]

### 17.6 Physical examples

**Example 17.9.** Population models

Suppose we have two countries with time varying populations \( x \) and \( y \). Suppose also that the natural growth rate in the countries is 2% and 2% respectively. In addition every year 3% of the country 1 moves to country 2 and 1% of country 2 moves to country 1.

Give a system of differential equations modeling this scenario. Assume initial populations of \( x(0) = 2 \) and \( y(0) = 2 \) (in units of one million). Solve the system and interpret the eigenvectors in terms of populations.

**answer:** We have

\[
\begin{align*}
x' &= 0.02x - 0.03x + 0.01y = -0.01x + 0.01y \\
y' &= 0.03x + 0.02y - 0.01y = 0.03x + 0.01y
\end{align*}
\]

\[
\Leftrightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -0.01 & 0.01 \\ 0.03 & 0.01 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

We solve using by finding eigenvalues and eigenvectors.

**Characteristic equation:**

\[
\begin{vmatrix} -0.01 - \lambda & 0.01 \\ 0.03 & 0.01 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda = 0.02, -0.02
\]

**Eigenvectors:** \( (A - \lambda I)v = 0 \).

\( \lambda_1 = 0.02: \begin{bmatrix} -0.03 & 0.01 \\ 0.03 & -0.01 \end{bmatrix} v = 0 \Rightarrow \text{take } v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \)

\( \lambda_2 = -0.02: \begin{bmatrix} 0.01 & 0.01 \\ 0.03 & 0.03 \end{bmatrix} v = 0 \Rightarrow \text{take } v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \)

The general solution is

\[
\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{0.02t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-0.02t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]
The initial conditions produce \( c_1 = 1 \) and \( c_2 = 1 \). So

\[
\begin{bmatrix}
x
\end{bmatrix}
= e^{0.02t}
\begin{bmatrix}
1 \\
3
\end{bmatrix}
+ e^{-0.02t}
\begin{bmatrix}
1 \\
-1
\end{bmatrix}
\]

Over time the \( e^{-0.02t} \) term will go to 0 and the populations will grow exponentially and remain in a ratio of \( x/y = 1/3 \).

Some eigenvectors may have negative entries and some eigenvalues may be negative or complex. However, any population vector is a combination of these pure modes.

**Example 17.10. Coupled springs.** Suppose we have two masses and springs configured as shown.

\[
\begin{align*}
\dot{x} &= -k_1 x + k_2 (y - x) \\
\dot{y} &= -k_2 (y - x) + f(t)
\end{align*}
\]

\( x \) is the displacement of \( m_1 \) from its equilibrium position.
\( y \) is the displacement of \( m_2 \) from its equilibrium position.

(\( x(t) \) is the amount that spring 2 is stretched is \( y - x \).)
\( f(t) \) is a time-varying force applied to \( m_2 \).

We can rearrange this to be

\[
\begin{align*}
\ddot{x} &= \frac{k_1 + k_2}{m_1} x + \frac{k_2}{m_2} y \\
\ddot{y} &= \frac{k_2}{m_1} x - \frac{k_2}{m_2} y + f(t)
\end{align*}
\]

The system is fourth-order because it consists of 2 second-order equations. You should think about how you would produce a companion system of 4 first order equations.

This system is illustrated by the applet (you have to set one of the spring constants to 0)

http://mathlets.org/mathlets/coupled-oscillators/

**Example 17.11. Salt tanks.** Suppose we have two tanks containing a salt solution.

Initially the volume of water in the tanks is \( V_1 \) and \( V_2 \) respectively. Pure water flows into tank from the outside at \( r_I \) liters/minute. Solution flows out of tank 2 at a rate of \( r_O \) liters/min. Solution is exchanged between the tanks, as shown, at the rates \( r_1 \) and \( r_2 \) in liters/min.

Suppose the rates and volumes are:

\( r_I = 20 \) (pure water), \( r_1 = 10 \), \( r_2 = 30 \), \( r_O = 20 \)
\( V_1 = 100 \) liters, \( V_2 = 200 \) liters.

Note that the flow rates are balanced so that \( V_1 \) and \( V_2 \) do not change.
Write a system of DEs modeling the amount of salt in each tank.

**Answer:** Let \( x \) be the grams of salt in tank 1, and \( y \) the grams of salt in tank 2.

Before starting let’s note that because pure water is being added all the salt will eventually be flushed out of the tanks, i.e., both \( x \) and \( y \) \( \to 0 \) in the long run. We should check our answer against these facts.

Now for the model:

\[
x' = \text{rate in} - \text{rate salt out (of tank 1)}.
\]

rate in = flow \cdot concentration = \( r_1 \cdot \frac{y}{V_2} = 10 \text{ l/min} \cdot \frac{g}{200 \text{ l}} = \frac{10}{200} \frac{y}{g/\text{min}} \).

rate out = \( r_2 \cdot \frac{x}{V_1} = \frac{30}{100} x \text{ g/min} \).

Thus: \( x' = -\frac{3}{10} x + \frac{1}{20} y \)

Likewise for \( y' \):

rate in = \( r_2 \cdot \frac{x}{V_2} \), rate out = \( (r_1 + r_O) \cdot \frac{y}{V_2} \)

So, \( y' = \frac{3}{10} x - \frac{3}{20} y \).

**Example 17.12.** A book has three axes of symmetry. It will spin nicely about these axes. (Although the one labeled \( a_1 \) is so unstable that it is difficult to actually spin the book around it.) These are the pure modes. Spinning in any other way is a combination of these pure modes. The axes are eigenvectors of the coefficient matrix for the system of differential equations modeling this system.