18 Fundamental matrix, matrix exponential

18.1 Goals

1. Be able to recognize a linear non-constant coefficient system of differential equations.
2. Know the definition and basic properties of a fundamental matrix for such a system.
3. Know the definition of the matrix exponential.
4. Be able to compute the matrix exponential from eigenvalues and eigenvectors.
5. Be able to use the matrix exponential as a fundamental matrix for a constant coefficient linear system.

18.2 Introduction

So far we have focused on homogeneous, constant coefficient linear systems. We now want to think about systems with input or with non-constant coefficients. So in this topic we will consider general linear systems of differential equations. That is, equations of the following form.

\[ \mathbf{x}' = A(t)\mathbf{x} \]  \hspace{1cm} \text{(homogeneous) (H)}

\[ \mathbf{x}' = A(t)\mathbf{x} + \mathbf{F}(t) \]  \hspace{1cm} \text{(inhomogeneous) (I)}

Here \( \mathbf{x}(t) \) is a vector valued function, e.g. \((x(t), y(t), z(t))^T\), \( A(t) \) is an \( n \times n \) matrix called the coefficient matrix and \( \mathbf{F}(t) \) is called input to the system.

As usual, solving the system means finding the unknown vector valued function \( \mathbf{x}(t) \).

The main point in this topic is to introduce the fundamental matrix \( \phi(t) \) for a linear system of DEs. This will allow us to state the essential properties of these systems in a concise and elegant way.

The fundamental matrix is available for any linear systems. For constant coefficient systems \( \mathbf{x}' = A\mathbf{x} \) we’ll use the diagonalization \( A = SAS^{-1} \) to define the matrix exponential \( e^{At} \) and see that this is a fundamental matrix of a special type.

18.3 Linearity/Superposition

As always, linear systems satisfy superposition principles. We restate them in the forms we like to use.

1. If \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \) are solutions to (H) then so is \( \mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 \)

Proof. \( \mathbf{x}' = c_1\mathbf{x}_1' + c_2\mathbf{x}_2' = c_1A\mathbf{x}_1 + c_2A\mathbf{x}_2 = A(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = A\mathbf{x} \).

2. If \( \mathbf{x}_h \) is a solution to (H) and \( \mathbf{x}_p \) is a solution to (I) then \( \mathbf{x} = \mathbf{x}_p + \mathbf{x}_h \) is also a solution to (I).
Proof. \( x' = x_p' + x_h' = Ax_p + F + Ax_h = A(x_p + x_h) + F = Ax + F. \)

3. If \( x_1' = Ax_1 + F_1 \) and \( x_2' = Ax_2 + F_2 \) then \( x_1 + x_2 \) satisfies \( x' = Ax + F_1 + F_2 \)
That is, superposition of inputs leads to superposition of outputs.
Proof. Just the same.

18.4 Existence and uniqueness theorem

As we've done for other types of equations, we state an existence and uniqueness theorem so that we can be sure that we have found all the solutions when we use the \( x(t) = x_p(t) + x_h(t) \) paradigm.

Consider the initial value problem:
\[
\begin{align*}
   x' &= A(t)x + F(t), & x(0) &= x_0 \\
\end{align*}
\]

The existence and uniqueness theorem says that there is exactly one solution to this equation.

**Theorem. (existence and uniqueness)** If \( A(t) \) and \( F(t) \) are continuous then there exists a unique solution to the equation (IVP).

The next example illustrates that this new version of the existence and uniqueness theorem agrees with our old version for second order linear equations.

**Example 18.1.** Consider the IVP \( x'' + tx' + t^2x = t^3; \quad x(0) = 1, \quad x'(0) = 3. \)

Converting this DE to a system using \( y = x' \) we get:
\[
\begin{bmatrix}
   x' \\
   y'
\end{bmatrix} = \begin{bmatrix}
   0 & 1 \\
   t^2 & -t
\end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ t^3 \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

More abstractly we can write this as: \( x' = Ax + F; \quad x(0) = (1, 3)^T \)
Since \( A(t) \) and \( F(t) \) are continuous the existence and uniqueness for systems says there is a unique solution to the system. Now, \( x(t) \) is the first entry in this solution so there is also a unique solution to the original IVP.

**Note.** Previously we had an existence and uniqueness theorem for ordinary differential equations which said exactly the same thing.

18.5 Fundamental matrix

This is an elegant bookkeeping technique which will make calculations and theorem statements much nicer. Consider the linear homogeneous system
\[
   x' = A(t)x \quad \text{(H)}
\]

Suppose it is an \( n \times n \) system and that we have \( n \) independent solutions \( x_1, \ldots, x_n \). We define the **fundamental matrix** as the matrix with columns \( x_1, \ldots, x_n \), i.e.
\[
   \Phi(t) = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \end{bmatrix}.
\]
Example 18.2. Consider the IVP \( x' = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} x, \ x(t_0) = b. \)

We know two independent solutions to this system are \( x_1 = e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ x_2 = e^{7t} \begin{bmatrix} 5 \\ 1 \end{bmatrix}. \) So the fundamental matrix is \( \Phi(t) = \begin{bmatrix} e^t & 5e^{7t} \\ -e^t & e^{7t} \end{bmatrix}. \)

Because multiplying \( \Phi \) by a vector gives a linear combination of the columns we can write the general solution can be written using \( \Phi(t) \):

\[
\begin{align*}
x &= c_1 x_1 + c_2 x_2 = c_1 \begin{bmatrix} e^t \\ -e^t \end{bmatrix} + c_2 \begin{bmatrix} 5e^{7t} \\ e^{7t} \end{bmatrix} = \Phi(t) \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.
\end{align*}
\]

(Make sure you recall why matrix multiplication is a linear combination of the columns of \( \Phi \).)

We can use this to find the solution to the IVP with initial conditions \( x(t_0) = b. \)

\[
\begin{align*}
x(t) &= \Phi(t) \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow \Phi(t_0) \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = b \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \Phi^{-1}(t_0) b.
\end{align*}
\]

This works provided \( \Phi^{-1}(t_0) \) exists –this is shown below.

18.5.1 Properties of \( \Phi \)

We have the following important properties of the fundamental matrix \( \Phi \).

1. \( \Phi'(t) = A(t) \Phi(t) \) i.e., \( \Phi \) satisfies (H).

2. If \( c \) is a column vector then \( \Phi(t) \cdot c = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n. \)

3. If \( A(t) \) is continuous then \( W(t) = |\Phi(t)| \neq 0 \) equivalently \( \Phi^{-1}(t) \) exists. (We call \( W(t) \) the Wronskian of \( x_1, \ldots, x_n \).)

Proof. (1) Before proving this we note the following property of matrix multiplication. If \( B \) has columns \( b_1, \ldots, b_n \) then

\[
AB = \begin{bmatrix} Ab_1 \ldots Ab_n \end{bmatrix}.
\]

You should make sure you understand this. (If it is confusing work out a simple numerical example with an eye to understanding this property.)

Now (1) follows easily from this property:

\[
\Phi'(t) = \begin{bmatrix} x'_1 & x'_2 & \ldots & x'_n \end{bmatrix} = \begin{bmatrix} Ax_1 & Ax_2 & \ldots & Ax_n \end{bmatrix} = A \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}
\]

The second equality above follows because the \( x_j \) are solutions to (H). The third equality is the property of matrix multiplication discussed just above.

(2) This is just a property of matrix multiplication.

(3) We will prove this by contradiction, i.e. we’ll assume that for some \( t_0, W(t_0) = 0 \) and show that this contradicts the existence and uniqueness theorem. So, suppose that
$W(t_0) = 0$. This implies that $\Phi(t_0)$ has a nontrivial null space. Let $c \neq 0$ be a nontrivial null vector. The contradiction is that now there are two solutions with $x(t_0) = 0$. That is, both

$$x_1(t) \equiv 0 \quad \text{and} \quad x_2(t) = \Phi(t)c$$

are 0 at $t = t_0$. This contradiction means that our assumption that $W(t_0) = 0$ must be false. QED

**Example 18.3.** Consider the system $x' = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} x$. Write down a fundamental matrix and show that its Wronskian is never 0.

**answer:** We have solved this system many times. Putting our two basic solutions as the columns of $\Phi$ we get:

Fundamental matrix $\Phi(t) = \begin{bmatrix} e^t & 5e^{7t} \\ -e^t & e^{7t} \end{bmatrix}$

Wronskian $W(t) = \det(\Phi(t)) = e^{8t} + 5e^{8t} = 6e^{8t}$ is never 0.

### 18.5.2 The Wronskian of $n$ solutions

In the above we assumed that the solutions were independent. Even if they are not we can still define the Wronskian: Suppose $x_1, \ldots, x_n$ are solutions to (H). We call the determinant $W(t) = \det(x_1 \ldots x_n)$ the Wronskian of these solutions. If $A(t)$ is continuous then the existence and uniqueness theorem implies:

(i) $W(t)$ is either always 0 or never 0.

(ii) $W(t) \neq 0 \iff x_1, \ldots, x_n$ are independent.

(iii) $W(t) \neq 0 \iff \Phi = \begin{pmatrix} x_1 & \ldots & x_n \end{pmatrix}$ is a fundamental matrix.

So in this case, we can use the Wronskian to test for independence.

**Example 18.4.** Consider $x'' + p(t)x' + q(t)x = 0$, with solutions $x_1, x_2$. Convert this to a system, give two solutions to the system and compute their Wronskian.

**answer:** The companion system is found by setting $y = x'$. Thus the solutions $x_1$ and $x_2$ of the ordinary differential equation become the solutions $x_1 = \begin{bmatrix} x_1 \\ x_1' \end{bmatrix}$ and $x_2 = \begin{bmatrix} x_2 \\ x_2' \end{bmatrix}$ of the companion system. Using the definition of the Wronskian we have

$$W(t) = \det \begin{bmatrix} x_1 & x_2 \\ x_1' & x_2' \end{bmatrix} = x_1x_2' - x_1'x_2.$$

### 18.6 Matrix Exponential

In 18.03 we use the exponential function all the time. It’s main property for us is that it helps us solve differential equations.

**Example 18.5.** Solve $x' = ax$
answer: \( x = x(0) e^{at} \).

We are going to define the matrix exponential. There are several ways to do this. Since this is differential equations class let’s define it as the solution to a DE. Then we will see various ways to compute it and to use it.

**Definition.** For any square matrix \( A \) the matrix exponential \( e^{At} \) is the matrix of functions that satisfies the initial value problem

\[
\frac{dB(t)}{dt} = A \cdot B(t), \quad B(0) = I.
\]

**Note.** We could also have defined \( e^{At} \) using the Taylor series for \( e^x \)

\[
e^{At} = I + tA + \frac{t^2 A^2}{2} + \frac{t^3 A^3}{3!} + \ldots
\]

Either definition gives the same answer.

We can now list several properties of the matrix exponential.

1. The initial value problem \( x' = Ax \) with initial value \( x(0) = b \) has solution \( e^{At}b \).

2. If \( A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \) then \( e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \).

3. If \( A = SAS^{-1} \) is the diagonalization of \( A \) then

\[
e^{At} = Se^{At}S^{-1}
\]

4. \( e^{At} \) is a fundamental matrix for the system \( x' = Ax \)

5. \( e^{A(s+t)} = e^{As}e^{At} \).

**Warning:** Because matrix multiplication does not commute you can’t count on the rules for exponents to hold. For example it is not generally true that \( e^{A+B} \) is the same as \( e^A e^B \). They are the same only in special circumstances.

**Proofs.** Here are proofs of these facts.

1. We need to verify that \( x(t) = e^{At}b \) satisfies the IVP. This follows directly from our definition of matrix exponential

\[
x'(t) = \frac{de^{At}b}{dt} = A e^{At}b = Ax(t).
\]

2. \[
\frac{d}{dt} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} \lambda_1 e^{\lambda_1 t} & 0 \\ 0 & \lambda_2 e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \Lambda e^{At} \text{ QED.}
\]

3. We need to show that \( \frac{d}{dt}S e^{At}S^{-1} = AS e^{At}S^{-1} \). The left-hand side of this equation is:

\[
\frac{d}{dt}S e^{At}S^{-1} = S \Lambda e^{At}S^{-1}.
\]

The right hand side of the equation is: \( AS e^{At}S^{-1} = SA e^{At}S^{-1} \).
The two sides are the same. QED

4. This is clear from (1).

5. This follows from the time invariance of the DE that $e^{At}$ satisfies.

**Example 18.6.** Let $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$ Solve the initial value problem $x' = Ax$, $x(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}^T$

**answer:** We know the answer is $x = e^{At} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

We can rewrite this as

$$x(t) = Se^{At}S^{-1} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{7t} & 0 \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{7t} & 0 \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} 8/6 \\ -22/6 \end{bmatrix}$$

$$= \frac{8}{6} e^{7t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} - \frac{22}{6} e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$