19 Variation of parameters; exponential inputs; Euler’s method

19.1 Goals

1. Be able to derive and apply the exponential response formula for constant coefficient linear systems with exponential input.

2. Be able to solve linear constant coefficient systems with sinusoidal input using complex replacement and the ERF.

3. Be able to use the variation of parameters formula to solve a (nonconstant) coefficient linear inhomogeneous system.

4. Be able to use Euler’s method to approximate the solution to a system of first order equations.

19.2 Introduction

We now turn our attention to inhomogeneous linear systems of the form

\[ x' = A(t)x + F(t). \] (I)

Here \( x \) is a column vector of (unknown) functions, \( A(t) \) is a square matrix and the input \( F(t) \) is a column vector.

The associated homogeneous equation is

\[ x' = A(t)x \] (H)

In this topic we will first consider the case where the coefficient matrix \( A \) is constant and the input \( F \) is exponential. Perhaps unsurprisingly, this will lead to the exponential response formula for a solution. As with ordinary differential equations, this will also allow us to find the response to sinusoidal input.

Next we will look at linear equations with arbitrary input. This will lead to the variation of parameters formula for the solution. This is a beautiful formula, it can be painful or difficult to apply. We will use it as a last resort to look for solutions to equations with nonconstant coefficients or unusual input.

19.3 Exponential response formula (ERF)

Exponential response formula. For a constant matrix \( A \) and a constant vector \( k \) the DE

\[ x' = Ax + e^{at}k \]

has a particular solution:

\[ x_p = -e^{at}(A - aI)^{-1}k \]
This formula is valid as long as $A - aI$ is invertible, i.e. as long as $a$ is not an eigenvalue of $A$.

**Proof.** Not surprisingly we discover this formula by the method of optimism. We try a solution of the form $x_p(t) = e^{at}v$, where $v$ is a constant vector.

Plug the guess into the DE and solve for $v$:

$$x_p' = ae^{at}v = e^{at}Av + e^{at}k \Rightarrow -(A - aI)v = k \Rightarrow v = -(A - aI)^{-1}k.$$ 

Thus, we have found a particular solution $x_p(t) = e^{at}v = -e^{at}(A - aI)^{-1}k$. QED

**Example 19.1.** Find the general solution to

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e^{2t} \\ 3e^{2t} \end{bmatrix}.$$ 

**answer:** For ease of notation we rewrite the equation as $\begin{bmatrix} x' \\ y' \end{bmatrix} = Ax + e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. The exponential response formula gives us a particular solution

$$x_p(t) = -e^{2t}(A - 2I)^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= -e^{2t} \begin{bmatrix} 4 & 5 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= -\frac{e^{2t}}{5} \begin{bmatrix} 0 & 5 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= -\frac{1}{5} e^{2t} \begin{bmatrix} 15 \\ -11 \end{bmatrix}$$

We know from previous topics that the general homogeneous equation is

$$x_h(t) = c_1e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2e^{7t} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

By superposition the general solution to the system is $x(t) = x_p(t) + x_h(t)$.

**Example 19.2.** Solve

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3e^{2t} \\ 5e^{3t} \end{bmatrix}.$$ 

**answer:** Write the input as $e^{2t} \begin{bmatrix} 3 \\ 0 \end{bmatrix} + e^{3t} \begin{bmatrix} 0 \\ 5 \end{bmatrix}$. Now you can solve the equation for each input term and then use superposition.

There are more examples in the next section.

### 19.4 Exponential response formula examples

**Example 19.3.** Find the general solution to

$$\begin{array}{c}
x' = 3x - y + e^{2t} \\
y' = 4x - y - e^{2t}
\end{array}$$
**answer:** In matrix form the equation is $\mathbf{x}' = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} \mathbf{x} + e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The exponential response formula tells us a particular solution is 

$$\mathbf{x}_p = -e^{2t}(A - 2I)^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -e^{2t} \begin{bmatrix} -3 & 1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = e^{2t} \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

We’ll let you verify the calculation of the inverse. Likewise we’ll let you find the homogeneous solution needed for the general solution.

**Example 19.4.** Find a particular solution to 

$$\begin{cases} x' = 3x - y + 3 \\ y' = 4x - y + 2 \end{cases}$$

**answer:** Note that we could use the exponential response formula, where the exponent is $a = 0$. Instead we’ll just try a constant solution and solve for its exact value.

Try $\mathbf{x} = \mathbf{v}$. Substitution into the DE gives $\mathbf{x}' = 0 = A\mathbf{v} + \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

So, $\mathbf{v} = -A^{-1} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = - \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$. That is $\mathbf{x}_p(t) = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$.

Again, we’ll let you verify the calculation of the inverse.

**Example 19.5.** Find a particular solution to 

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \cos(t) \\ 0 \end{bmatrix}.$$ 

**answer:** To use the exponential response formula we first need to use complex replacement. The complexified equation is 

$$\mathbf{z}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{z} + e^{it} \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where $\mathbf{x} = \text{Re}(\mathbf{z})$.

Now we set up the exponential response formula

$$(A - iI)^{-1} = \begin{bmatrix} 1 - i & 2 \\ 2 & 1 - i \end{bmatrix}^{-1} = \frac{1}{-2i - 4} \begin{bmatrix} 1 - i & -2 \\ -2 & 1 - i \end{bmatrix}.$$

So, 

$$\mathbf{z}_p = -e^{it}(A - iI)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\frac{1}{2i - 4} e^{it} \begin{bmatrix} 1 - i \\ -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{-2i + 4} e^{it} \begin{bmatrix} 1 - i \\ -2 \end{bmatrix}.$$ 

To find the real part of $\mathbf{z}_p$ we work in polar coordinates. First we write the various complex numbers in polar form:

$$2i + 4 = 2\sqrt{5}e^{i\phi_1}, \text{ where } \phi_1 = \text{Arg}(2i + 4) = \tan^{-1}(.5) \text{ in the first quadrant.}$$

Likewise $1 - i = \sqrt{2}e^{i\phi_2}$, with $\phi_2 = -\pi/4$.

So 

$$\mathbf{z}_p = -e^{it} \left[ \frac{\sqrt{2}e^{i\phi_2}}{2\sqrt{5}e^{i\phi_1}} \right] = -\frac{1}{2\sqrt{5}} \left[ \begin{bmatrix} \sqrt{2}e^{i(t + \phi_2 - \phi_1)} \\ -2e^{i(t - \phi_1)} \end{bmatrix} \right].$$

Taking the real part:

$$\mathbf{x}_p = \text{Re} (\mathbf{z}_p) = \begin{bmatrix} \sqrt{2} \cos(t + \phi_2 - \phi_1) \\ -2 \cos(t - \phi_1) \end{bmatrix}.$$
Here is the same calculation in rectangular coordinates. I think the arithmetic is more error prone and the answer is harder to interpret.

\[
\frac{1}{2i + 4} \begin{bmatrix} 1 - i \\ -2 \end{bmatrix} = \frac{4 - 2i}{20} \begin{bmatrix} 1 - i \\ -2 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 1 - 3i \\ -4 + 2i \end{bmatrix}.
\]

So,

\[
z_p = \frac{1}{10} \left( \cos(t) + i \sin(t) \right) \begin{bmatrix} 1 - 3i \\ -4 + 2i \end{bmatrix} = \frac{1}{10} \begin{bmatrix} \cos(t) + 3 \sin(t) + i(\sin(t) - 3 \cos(t)) \\ -4 \cos(t) - 2 \sin(t) + i(-4 \sin(t) + 2 \cos(t)) \end{bmatrix}
\]

Thus,

\[
x_p(t) = \text{Re}(z_p(t)) = \frac{1}{10} \left[ \begin{bmatrix} \cos(t) + 3 \sin(t) \\ -4 \cos(t) - 2 \sin(t) \end{bmatrix} \right].
\]

19.5 Variation of parameters

For the general, not necessarily constant coefficient, linear inhomogeneous system (I) we cannot use constant coefficient techniques like the ERF. For those cases where the ERF doesn’t apply we can try to use the variation of parameters formula. Since it involves integration, matrix inverses and matrix multiplication it is our last choice when trying to solve an equation. Nonetheless, sometimes it’s the only method available. In addition, the derivation of the formula is really very pretty.

Suppose we have a fundamental matrix \( \Phi(t) \) for the homogeneous linear equation

\[
x'(t) = A(t)x
\]

(II)

Remember this means that \( \Phi \) has columns which are independent solutions to (II).

19.6 Variation of parameters formula

Now suppose we want to solve

\[
x' = A(t)x + F(t).
\]

Theorem. The general solution to equation (I) is given by the variation of parameters formula

\[
x(t) = \Phi(t) \cdot \left( \int \Phi(t)^{-1} \cdot F(t) \, dt + C \right).
\]

Proof. (Remember this.) We will use a form of the method of optimism to derive this formula.

We know the general homogeneous solution is \( x(t) = \Phi(t) \cdot c \) for a constant vector \( c \). The vector \( c \) is called a parameter. Variation of parameters is an old-fashioned way of saying let’s optimistically make it a (dependent) variable \( u(t) \). So, we try a solution of the form \( x(t) = \Phi(t) \cdot u(t) \). The function \( u(t) \) is unknown, we substitute our guess into (I) and see what \( u(t) \) needs to be:

\[
\Phi' \cdot u + \Phi \cdot u' = A\Phi \cdot u + F
\]

So, (don’t forget \( \Phi' = A\Phi \))

\[
A\Phi \cdot u + \Phi \cdot u' = A\Phi \cdot u + F \implies \Phi \cdot u' = F.
\]
This last equation is easy to solve

\[ u' = \Phi^{-1} \cdot F \Rightarrow u(t) = \int \Phi^{-1}(t) \cdot F(t) \, dt + C. \]

We take this formula for \( u(t) \) and use it in our trial solution

\[ x(t) = \Phi(t) \cdot u(t) = \Phi(t) \cdot \left( \int \Phi(t)^{-1} \cdot F(t) \, dt + C \right). \]

QED.

**Remark.** Note that the variation of parameters formula assumes you know the general homogeneous solution. It gives no help in finding this solution. (For examples we can use constant coefficient systems where we know how to find the homogeneous solution using eigenvalues and eigenvectors.)

**Example 19.6.** Use the variation of parameters formula to solve

\[ x' = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} e^t \\ 5e^{7t} \end{bmatrix}. \]

**Note.** We retiterate that using the ERF is the preferred method of solving this equation. We use the variation of parameters formula here for practice.

**answer:** Let’s introduce some notation to save typing: \( A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}, \ F = \begin{bmatrix} 1 \\ t \end{bmatrix}. \)

We know the fundamental matrix from an earlier example: \( \Phi(t) = \begin{bmatrix} e^t & 5e^{7t} \\ -e^t & e^{7t} \end{bmatrix}. \) So,

\[ \Phi^{-1}(t) = \frac{e^{-8t}}{6} \begin{bmatrix} e^{7t} & -5e^{7t} \\ e^t & e^t \end{bmatrix}. \]

Calculating with the variation of parameters we get

\[ x = \Phi(t) \int \Phi^{-1}(t) \cdot F(t) \, dt \]

\[ = \Phi(t) \int \frac{1}{6} \begin{bmatrix} 1 - 5e^{4t} \\ e^{-6t} + e^{-2t} \end{bmatrix} dt \]

\[ = \frac{1}{6} \Phi(t) \left[ -\frac{t}{6} e^{-6t} + c_1 - \frac{5}{6} e^{-2t} + c_2 \right] \]

\[ = \frac{1}{6} \begin{bmatrix} te^t - \frac{5}{4} e^{5t} - \frac{5}{2} e^{5t} + c_1 e^t + 5c_2 e^{7t} \\ -te^t + \frac{5}{2} e^{5t} - \frac{1}{2} e^{5t} - c_1 e^t + c_2 e^{7t} \end{bmatrix} \]

\[ = \frac{1}{6} \begin{bmatrix} te^t - \frac{5}{4} e^{5t} + 3/4 e^{5t} - 1/6 e^{5t} + c_1 e^t + c_2 e^{7t} \end{bmatrix}. \]

Notice the homogeneous solution appearing with the constants of integration.
19.6.1 Definite integral version of variation of parameters

The equation (I) with initial condition $x(t_0) = b$ has definite integral solution

$$x(t) = \Phi(t) \left( \int_{t_0}^{t} \Phi^{-1}(u) \cdot F(u) \, du + C \right) \text{ where } C = \Phi^{-1}(t_0) \cdot b.$$

19.7 Euler’s method

Euler’s method works without change for systems.

Consider a first order system: $\dot{x} = F(x, t), \quad x(t_0) = x_0$.

Using stepsize $h$ we have the same algorithm as for ordinary DE’s.

$m = F(x_n, t_n) \Rightarrow x_{n+1} = x_n + h \cdot m, \quad t_{n+1} = t_n + h$.

Again, just like for ordinary DE’s, there are other, better, algorithms for choosing $m$ or varying $h$.

**Example 19.7.** Consider $\begin{bmatrix} x' \\ y' \end{bmatrix} = t \begin{bmatrix} y \\ x \end{bmatrix}, \quad x(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Let $x = \begin{bmatrix} x \\ y \end{bmatrix}$ and use $h = 0.5$ to estimate $x(2)$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$t_n$</th>
<th>$x_n$</th>
<th>$m = F(x_n, t_n)$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1.0</td>
<td>$\begin{bmatrix} 1 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \ 1 \end{bmatrix}$</td>
</tr>
<tr>
<td>1</td>
<td>1.5</td>
<td>$\begin{bmatrix} 1 \ .5 \end{bmatrix}$</td>
<td>$.75$ $\begin{bmatrix} .5 \ 1.5 \end{bmatrix}$</td>
</tr>
<tr>
<td>2</td>
<td>2.0</td>
<td>$\begin{bmatrix} 1.375 \ 1.25 \end{bmatrix}$</td>
<td></td>
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