22 Fourier series introduction: continued

22.1 Goals

1. Be able to compute the Fourier coefficients of even or odd periodic function using the simplified formulas.
2. Be able to write and graph the function a Fourier series converges to.
3. Be able to determine the decay rate of the coefficients of a Fourier series.
4. Be able to predict the decay rate of the Fourier coefficients based on how smooth the original function is.

22.2 Introduction

In this topic we continue our introduction to Fourier series. We start by looking at some tricks for computing Fourier coefficients. Then we will talk about more conceptual notions, including the convergence properties of Fourier series and the decay rate of Fourier coefficients. At the end we will look at the orthogonality relations which explain the our formulas for Fourier coefficients.

22.3 Calculation tricks: even and odd functions

22.3.1 Even and odd functions

A function is an even function if \( f(-t) = f(t) \) for all \( t \).

- The graph of an even function is symmetric about the \( y \)-axis.

\[
\begin{align*}
\text{Graphs of some even functions} \\
1, t^2, t^4, \ldots, \cos(\omega t).
\end{align*}
\]

- Examples of even functions: 1, \( t^2 \), \( t^4 \), \ldots, \( \cos(\omega t) \). In general, even functions are built out of even powers of \( t \). Note that, the power series for \( \cos(\omega t) \) has only even powers.
By symmetry we have the following key integration fact for even functions:

\[ \int_{-L}^{L} f(t) \, dt = 2 \int_{0}^{L} f(t) \, dt \quad \text{for any even } f(t). \]

A function is an odd function if \( f(-t) = -f(t) \) for all \( t \).

- The graph of an odd function is symmetric about the origin.

![Graphs of some odd functions](image)

- Examples of odd functions: \( t, t^3, t^5, \ldots, \sin(\omega t) \). In general, odd functions are built out of odd powers of \( t \). Note that, the power series for \( \sin(\omega t) \) has only odd powers.

- By symmetry we have the following key integration fact for odd functions:

\[ \int_{-L}^{L} f(t) \, dt = 0 \quad \text{for any odd } f(t). \]

**Products of even and odd functions**

We give the rules in a kind of short-hand. You can remember these rules by thinking about powers of \( t \), e.g. \( t^4 \cdot t^7 = t^{11} \), so even \( \cdot \) odd is odd.

- even \( \cdot \) even = even
- odd \( \cdot \) odd = even
- odd \( \cdot \) even = odd

### 22.3.2 Fourier coefficients of even and odd functions

- If \( f(t) \) even then \( b_n = 0 \) and \( a_n = \frac{2}{L} \int_{0}^{L} f(t) \cos\left(\frac{n\pi}{L} t\right) \, dt \).

- If \( f(t) \) is odd then \( a_n = 0 \), and \( b_n = \frac{2}{L} \int_{0}^{L} f(t) \sin\left(\frac{n\pi}{L} t\right) \, dt \).

**Reason.** If \( f(t) \) is even then by the multiplication rules for even functions \( f(t) \cos(\omega t) \),

\[ a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi}{L} t\right) \, dt = \frac{2}{L} \int_{0}^{L} f(t) \cos\left(\frac{n\pi}{L} t\right) \, dt. \]

Likewise, \( f(t) \sin(\omega t) \) is odd, so \( b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi}{L} t\right) \, dt = 0 \).

**Example 22.1.** In the last topic we met the period \( 2\pi \) square wave, which over one period has the formula

\[ \text{sq}(t) = \begin{cases} 
-1 & \text{for } -\pi \leq t < 0 \\
1 & \text{for } 0 \leq t < \pi.
\end{cases} \]
Since the period is $2\pi$ we have $L = \pi$. Since $\text{sq}(t)$ is odd, we know that $a_n = 0$ and

$$b_n = \frac{2}{\pi} \int_0^\pi \text{sq}(t) \sin(nt) \, dt = \frac{2}{\pi} \int_0^\pi \sin(nt) \, dt = -\frac{2}{n\pi} \cos(nt) \bigg|_0^\pi = \begin{cases} \frac{4}{n\pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}.$$  

We have found the Fourier series for $\text{sq}(t)$:

$$\text{sq}(t) = \sum_{n=1}^{\infty} b_n \sin(nt) = 4 \pi \left( \sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots \right)$$

**Example 22.2.** Triangle wave function (also called the continuous sawtooth function). Let $f(t)$ have period $2\pi$ and $f(t) = |t|$ for $-\pi \leq t \leq \pi$.

Since $f(t)$ is an even function we know that $b_n = 0$ and for $n \neq 0$ we have

$$a_n = \frac{1}{\pi} \int_{-\pi}^\pi |t| \cos(nx) \, dx = \frac{2}{\pi} \int_0^\pi x \cos(nx) \, dx$$

$$= \frac{2}{\pi} \left[ \frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} \right]^\pi_{-\pi} = \frac{2}{n^2\pi} ((-1)^n - 1) = \begin{cases} \frac{-4}{n^2\pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}.$$  

As always we compute $a_0$ separately: $a_0 = \frac{1}{\pi} \int_{-\pi}^\pi |t| \, dx = \frac{2}{\pi} \int_0^\pi t \, dt = \frac{\pi}{2}$.

Thus we have the Fourier series for $f(t)$:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) = \frac{\pi}{2} - \frac{4}{\pi} \left( \cos(t) + \frac{\cos(3t)}{3^2} + \frac{\cos(5t)}{5^2} + \cdots \right)$$
22.4 Summing Fourier series

We can use the sum of a finite number of terms from a Fourier series to approximate the original function. The applet [http://web.mit.edu/jorloff/www/jmoapplets/html5/fourierapproximation.html](http://web.mit.edu/jorloff/www/jmoapplets/html5/fourierapproximation.html) illustrates this. In the following sections we will bring out the following key points:

- The first few terms of the Fourier series approximate the shape of the function, not necessarily the value of the function at any one point.
- At points of continuity the Fourier series converges to the original function.
- The smoother the function the faster the Fourier series converges to the function.
- At jumps in the graph, no matter how many terms you use the Fourier series always overshoots the graph near that point.
- The smoother the function the faster the Fourier series converges to the function.

22.5 Convergence of Fourier series

**Piecewise smooth:** The period $2L$ function $f(t)$ is called piecewise smooth if there are only a finite number of points $0 \leq t_1 < t_2 < \ldots < t_n \leq 2L$ where $f(t)$ is not differentiable and at each of these points the left and righthand limits

$$f(t_i^+) = \lim_{t \to t_i^+} f(t) \quad \text{and} \quad f(t_i^-) = \lim_{t \to t_i^-} f(t)$$

exist (although they might not be equal).

In short, a function is piecewise smooth if it is smooth except at a discrete set of points where it has jump discontinuities.

Here is our main theorem about convergence of Fourier series. We will not prove it in 18.03.

**Theorem:** If $f(t)$ is piecewise smooth and periodic then the Fourier series for $f$

1. converges to $f(t)$ at values of $t$ where $f$ is continuous
2. converges to the average of $f(t^-)$ and $f(t^+)$ at values of $t$ where $f(t)$ has a jump discontinuity.

**Example 22.3. Square wave.** The square wave in the example above has jump discontinuities. No matter how we specify the endpoint behavior of $\text{sq}(t)$ the Fourier series converge to 0 at the discontinuities.
Example 22.4. The triangle wave in the example above is continuous so its Fourier series converges to the original function $f(t)$.

Example 22.5. We give one more graphical example. Here the original function has discontinuities—admittedly somewhat artificial. Since the left and righthand limits are the same at each discontinuity the Fourier series is continuous.

Example 22.6. The Fourier coefficients of $\text{sq}(t)$ are

$$a_n = 0 \quad \text{and} \quad b_n = \begin{cases} 
\frac{4}{(n\pi)} & \text{for } n \text{ odd} \\
0 & \text{for } n \text{ even}
\end{cases}$$

We say these coefficients decay like $1/n$.

Example 22.7. The triangle wave looked at above has Fourier coefficients

$$b_n = 0 \quad \text{and} \quad a_n = \begin{cases} 
-\frac{4}{(n^2\pi)} & \text{for } n \text{ odd} \\
0 & \text{for even } n \neq 0
\end{cases}$$

So these coefficients decay like $1/n^2$.

Example 22.8. The coefficients $a_n = 1/(n + n^2)$ decay like $1/n^2$.

Example 22.9. If a Fourier series has $a_n = 1/n$ and $b_n = 1/n^2$ we say the Fourier coefficients decay like $1/n$. That is, the decay rate is the slower of the two decay rates.

Example 22.10. The function $f(t) = 3\cos(t) + 5\cos(2t)$ is a already given as a (finite) Fourier series. The coefficients are $a_0 = 0$, $a_1 = 3$, $a_2 = 5$, $a_3 = 0$, $a_4 = 0$, . . . We say these coefficients have infinite decay rate. That is the decay faster than $1/n^p$ for any $p$.

22.5.2 Important heuristics

- If a function has a jump discontinuity then its Fourier coefficients decay like $1/n$, e.g. the square wave.
• If a function has a corner then its Fourier coefficients decay like $\frac{1}{n^2}$, e.g. the triangle wave
• A smooth function has Fourier coefficients that decay like $\frac{1}{n^3}$ or faster.
• The smoother the function, the faster the coefficients decay.

22.6 Gibbs’ phenomenon

Non-local nature of Fourier series
They approximate over the whole interval, not just near 0. (Analogy: least squares fit of data points.)

**Gibbs phenomenon** For the square wave the Fourier transform fits well at points of continuity by there is *always* about a 9% error near points of discontinuity – no matter how many terms of the series you use. I won’t give good physical examples for 18.03, but it is extremely important for many things, e.g. digital filtering of signals.

22.7 Orthogonality relations

This will probably not be talked about in class.

22.7.1 The orthogonality relations

The key to the integral formulas for Fourier coefficients are the orthogonality relations. These are the following integral formulas that say certain trigonometric integrals are either 0 or 1. The term orthogonality refers to the fact that if you have a set of mutually orthogonal unit vectors in a vector space then the dot product of any two
is either 0 (if they’re different) or 1 (if they’re the same).

\[
\frac{1}{L} \int_{-L}^{L} \cos \left( \frac{n\pi}{L} t \right) \cos \left( \frac{m\pi}{L} t \right) \, dt = \begin{cases} 
1 & n = m \neq 0 \\
0 & n \neq m \\
2 & n = m = 0
\end{cases}
\]

\[
\frac{1}{L} \int_{-L}^{L} \sin \left( \frac{n\pi}{L} t \right) \cos \left( \frac{m\pi}{L} t \right) \, dt = 0
\]

\[
\frac{1}{L} \int_{-L}^{L} \sin \left( \frac{n\pi}{L} t \right) \sin \left( \frac{m\pi}{L} t \right) \, dt = \begin{cases} 
1 & n = m \neq 0 \\
0 & n \neq m
\end{cases}
\]

**Proof.** We have two methods to do this. We will carry out the first, but only mention the second.

Method 1: Use the following trig. identities

\[
\cos(\alpha) \cos(\beta) = \frac{(\cos(\alpha + \beta) + \cos(\alpha - \beta))}{2}
\]

\[
\sin(\alpha) \cos(\beta) = \frac{(\sin(\alpha + \beta) + \sin(\alpha - \beta))}{2}
\]

\[
\sin(\alpha) \sin(\beta) = \frac{(\cos(\alpha - \beta) - \cos(\alpha + \beta))}{2}
\]

Method 2: Use \(\cos(at) = \frac{e^{iat} + e^{-iat}}{2}\) etc.

Using method 1 we get the following if \(n \neq m\):

\[
\frac{1}{L} \int_{-L}^{L} \cos \left( \frac{n\pi}{L} t \right) \cos \left( \frac{m\pi}{L} t \right) \, dt = \frac{1}{L} \int_{-L}^{L} \cos \left( \frac{(n+m)\pi}{L} t \right) + \cos \left( \frac{(n-m)\pi}{L} t \right) \, dt
\]

\[
= \frac{1}{2L} \left[ \sin \left( \frac{(n+m)\pi}{L} t \right)/L + \sin \left( \frac{(n-m)\pi}{L} t \right)/L \right]_{-L}^{L}
\]

\[
= 0.
\]

The last equality is easy to see since every term is 0 when \(t = \pm L\).

The case \(n = m\) is special because then \(n - m = 0\). It is easy to use the trig. identity to see that the integral in this case is 1. All the other orthogonality relations are proved in a similar fashion.

### 22.7.2 Using orthogonality relations to show the formula for Fourier coefficients

The orthogonality relations allow us to see that if \(f(t)\) is written as a Fourier series then the coefficients must be given by the integral formulas we’ve been using.
So, suppose $f(t)$ has Fourier series:

$$f(t) = \frac{a_0}{2} + a_1 \cos \left( \frac{\pi}{L} t \right) + a_2 \cos \left( \frac{2\pi}{L} t \right) + \cdots + b_1 \sin \left( \frac{\pi}{L} t \right) + b_2 \sin \left( \frac{2\pi}{L} t \right) + \cdots$$

Then for $n > 0$

$$\frac{1}{L} \int_{-L}^{L} f(t) \cos \left( \frac{n\pi}{L} t \right) \, dt = \frac{1}{L} \int_{-L}^{L} \left[ \frac{a_0}{2} \cos \left( \frac{n\pi}{L} t \right) + a_1 \cos \left( \frac{\pi}{L} t \right) \cos \left( \frac{n\pi}{L} t \right) + \cdots + b_1 \sin \left( \frac{\pi}{L} t \right) \cos \left( \frac{n\pi}{L} t \right) + b_2 \sin \left( \frac{2\pi}{L} t \right) \cos \left( \frac{n\pi}{L} t \right) + \cdots \right] \, dt$$

Now we can apply the orthogonality relations to each term. All of them are 0, except the term with $a_n \cos \left( \frac{n\pi}{L} t \right) \cos \left( \frac{n\pi}{L} t \right)$ which, again by the orthogonality relations, integrates to $a_n$. Thus $\frac{1}{L} \int_{-L}^{L} f(t) \cos \left( \frac{n\pi}{L} t \right) \, dt = a_n$. Which is exactly the formula for the Fourier coefficient. The formulas for $a_0$ and $b_n$ are found in the same way.

### 22.8 Hearing a musical triad: C-E-G

Here is a simplified Fourier-centric view of how humans hear sound.

Sound reaches your ear as a pressure wave. For example

$$f(t) = a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t) + \cdots$$

Do the ears do Fourier analysis?

Answer: Yes! The ear contains hair-like structures called stereocilia. These are different sizes and, so, resonate at different frequencies. As they vibrate they stimulate nerves which then send signals to the brain. Thus, for each frequency in the pressure wave the brain is getting a signal from the nerves attached to the stereocilia which vibrate at that frequency. The greater the amplitude in the input wave the greater the amplitude of the signal sent to the brain.

Does the brain do Fourier synthesis?

Answer: Yes! It is up to the brain to combine all the nerve signals at different frequencies into a single signal which it then interprets.