28 Linearization of nonlinear systems

28.1 Nonlinear Systems

A general first order autonomous \((2 \times 2)\) system has the following form

\[
\begin{align*}
x' &= f(x, y) \\
y' &= g(x, y)
\end{align*}
\]

Vector Field: This defines a vector field \((f(x, y), g(x, y))\) that attaches the velocity vector to each point \((x, y)\) in the phase plane.

By definition a critical point is one where \(x' = 0\) and \(y' = 0\). That is, it is a point \((x_0, y_0)\) where

\[f(x_0, y_0) = 0, \quad \text{and} \quad g(x_0, y_0) = 0.\]

Equivalently, it is an equilibrium solution \(x(t) = x_0, \ y(t) = y_0\). This is a solution whose trajectory is a single point.

28.2 Linearization around a critical point

We’ll start by presenting the method of linearization to sketch the phase portrait. First we’ll use it in an example. After that we’ll justify the method.

Jacobian. At a critical point \((x_0, y_0)\) of the system \(\text{1}\) we define the Jacobian by

\[
J(x_0, y_0) = \begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}.
\]

This gives the linearization around the critical point \((x_0, y_0)\)

\[
\begin{bmatrix} u \\ v \end{bmatrix}' = J(x_0, y_0) \begin{bmatrix} u \\ v \end{bmatrix}
\]

In general, the nonlinear system behaves like the linearized one. (We will learn the exceptions later.) That is, if we center our \(uv\)-axes on \((x_0, y_0)\) then the linear vector field near the \(uv\) origin approximates the nonlinear field near \((x_0, y_0)\)
Near a critical point the nonlinear system, is approximately linear.

**Example 28.1.** Find the critical points for the following system.

\[
\begin{align*}
x' &= 14x - \frac{1}{2}x^2 - xy \\
y' &= 16y - \frac{1}{2}y^2 - xy
\end{align*}
\]

**answer:** We solve the equations \(x' = 0\), \(y' = 0\).

\[
\begin{align*}
x' &= x \left(14 - \frac{1}{2}x - y\right) = 0 \Rightarrow x = 0 \text{ or } 14 - \frac{1}{2}x - y = 0 \\
y' &= y \left(16 - \frac{1}{2}y - x\right) = 0 \Rightarrow y = 0 \text{ or } 16 - \frac{1}{2}y - x = 0.
\end{align*}
\]

Looking at the product for \(x'\) we see \(x' = 0\) when \(x = 0\) or \(14 - x/2 - y = 0\). Likewise, \(y' = 0\) when \(y = 0\) or \(16 - y/2 - x = 0\). This leads to four sets of equations for critical points.

\[
\begin{align*}
\{x = 0\} & \quad \{x = 0\} & \quad \{14 - x/2 - y = 0\} & \quad \{14 - x/2 - y = 0\} \\
\{y = 0\} & \quad \{16 - y/2 - x = 0\} & \quad \{y = 0\} & \quad \{16 - y/2 - x = 0\}
\end{align*}
\]

The first three sets are easy to solve by inspection. The fourth requires a small computation. We get the following four critical points:

\((0, 0), (0, 32), (28, 0), (12, 8)\).

**Example 28.2.** (Continued from previous example.) Linearize the system at each of the critical points and determine the type of the linearized critical point.

**answer:** The linearized system at \((x_0, y_0)\) is \[
\begin{bmatrix}
\begin{bmatrix} u' \\
v' 
\end{bmatrix}
\end{bmatrix} = J(x_0, y_0) \begin{bmatrix}
\begin{bmatrix} u \\
v 
\end{bmatrix}
\end{bmatrix}.
\]

First we compute the Jacobian:

\[
J(x, y) = \begin{bmatrix} 14 - x - y & -x \\ -y & 16 - y - x \end{bmatrix}
\]

Next we look at each of the critical points in turn.

Critical point \((0, 0)\):

\[
J(0, 0) = \begin{bmatrix} 14 & 0 \\ 0 & 16 \end{bmatrix}; \text{ eigenvalues } 14, 16; \text{ eigenvectors } \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

This is a source node. Its sketch on \(uv\)-axes is shown in the left-most figure below.
Critical point \((0, 32)\):

\[
J(0, 32) = \begin{bmatrix}
-18 & 0 \\
-32 & -16
\end{bmatrix}; \text{ eigenvalues } -18, -16; \text{ corresponding eigenvectors } \begin{bmatrix} 1 \\ -16 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

This is a sink node. Its sketch is shown in the 'Sink node 1' figure above.

Critical point \((28, 0)\):

\[
J(28, 0) = \begin{bmatrix}
-14 & -28 \\
0 & -12
\end{bmatrix}, \text{ eigenvalues } -14, -12; \text{ corresponding eigenvectors } \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -14 \\ 1 \end{bmatrix}
\]

This is a sink node. Its sketch is shown in the 'Sink node 2' figure above.

Critical point \((12, 8)\):

\[
J(12, 8) = \begin{bmatrix}
-6 & -12 \\
-8 & -4
\end{bmatrix}; \text{ eigenvalues } -5 \pm \sqrt{97} \approx -15, 5.
\]

Eigenvectors: For \(\lambda = -5 - \sqrt{97}\): \[
\begin{bmatrix} 1 + \sqrt{97} \\ 8 \end{bmatrix} \approx \begin{bmatrix} 11 \\ 8 \end{bmatrix}
\]

For \(\lambda = -5 + \sqrt{97}\): \[
\begin{bmatrix} 1 - \sqrt{97} \\ 8 \end{bmatrix} \approx \begin{bmatrix} -9 \\ 8 \end{bmatrix}
\]

This is a saddle. Its sketch is shown in the 'Saddle' figure above.

To make a rough sketch of the nonlinear system’s phase portrait we: 1. Sketch the phase portrait near each critical point, using the linearization.
2. Connect these sketches together in a consistent manner.

We do this below and compare it with the sketch made by a Matlab program called PPlane.

![Hand sketch of the phase plane.](image)

![PPlane plot of the phase plane.](image)

### 28.2.1 Justification for using linearization

We’ll go through this in detail. One key fact is that the change of variables \(u = x - x_0, v = y - y_0\) puts the \(uv\) origin at \((x_0, y_0)\).

We will use the tangent plane, i.e. linear approximations of \(f\) and \(g\). You might recall this from 18.02. If not, notice that is just a multivariable version of the single variable approximation

\[
f(x) \approx f(x_0) + f'(x_0)\Delta x,
\]
where $\Delta x = x - x_0$.

For small changes $(x - x_0) = \Delta x$ and $(y - y_0) = \Delta y$ the tangent plane approximations for $f$ and $g$ near $(x_0, y_0)$ are

\[
\begin{align*}
    f(x, y) &\approx f(x_0, y_0) + f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y \\
    g(x, y) &\approx g(x_0, y_0) + g_x(x_0, y_0) \Delta x + g_y(x_0, y_0) \Delta y
\end{align*}
\]

Now let $u = x - x_0 = \Delta x$ and $v = y - y_0 = \Delta y$. Note two things.

1. This puts the origin of the $uv$-plane at $(x_0, y_0)$.
2. As functions of $t$: $u' = x'$, $v' = y'$ (since $x_0$ and $y_0$ are constants).

Using $u$ and $v$

\[
\begin{align*}
    f(x_0 + u, y_0 + v) &\approx f(x_0, y_0) + f_x(x_0, y_0) u + f_y(x_0, y_0) v \\
    g(x_0 + u, y_0 + v) &\approx g(x_0, y_0) + g_x(x_0, y_0) u + g_y(x_0, y_0) v
\end{align*}
\]

Writing these in matrix form we see the Jacobian appear:

\[
\begin{bmatrix}
    f(x_0 + u, y_0 + v) \\
    g(x_0 + u, y_0 + v)
\end{bmatrix} \approx \begin{bmatrix}
    f(x_0, y_0) \\
    g(x_0, y_0)
\end{bmatrix} + \begin{bmatrix}
    f_x(x_0, y_0) & f_y(x_0, y_0) \\
    g_x(x_0, y_0) & g_y(x_0, y_0)
\end{bmatrix}
\begin{bmatrix}
    u \\
    v
\end{bmatrix}
\]

If $(x_0, y_0)$ is a critical point the first term on the right is 0, i.e

\[
\begin{bmatrix}
    f(x_0 + u, y_0 + v) \\
    g(x_0 + u, y_0 + v)
\end{bmatrix} \approx J(x_0, y_0)
\begin{bmatrix}
    u \\
    v
\end{bmatrix}.
\]

Now, $u = x - x_0$ can be rewritten $x = x_0 + u$. Remembering that $u' = x'$, $v' = y'$ we put everything together as

\[
\begin{bmatrix}
    u' \\
    v'
\end{bmatrix} = \begin{bmatrix}
    x' \\
    y'
\end{bmatrix} = \begin{bmatrix}
    f(x_0 + u, y_0 + v) \\
    g(x_0 + u, y_0 + v)
\end{bmatrix} \approx J(x_0, y_0)
\begin{bmatrix}
    u \\
    v
\end{bmatrix}
\]

Using just the first and last terms from the above gives the linearization formula

\[
\begin{bmatrix}
    u' \\
    v'
\end{bmatrix} \approx J(x_0, y_0)
\begin{bmatrix}
    u \\
    v
\end{bmatrix}.
\]

This is a linear system with coefficient matrix $J(x_0, y_0)$. We call it the linearization of the system around the critical point.