29 Structural Stability

Structural stability is about the type of system not the type of critical point of the system.

Consider the following two scenarios.

Scenario 1. You have an apparatus modeled by a constant coefficient linear system \( x' = Ax \). You are experimentally able to measure the entries of the matrix \( A \) to two decimal places of accuracy. You are not surprised when your experiments reveal \( A = \begin{bmatrix} 6.00 & 5.00 \\ 1.00 & 2.00 \end{bmatrix} \).

So the eigenvalues of your system are 7.00 and 1.00.

You have experimentally determined that the equilibrium at the origin is a nodal source, which is unstable. But we have to take into account the possibility (really, guarantee) of measurement error. Each of your matrix entries might be off by as much as 0.005. Thus, the eigenvalues are also only approximately correct.

Nonetheless, with such small errors, the eigenvalues are both guaranteed to be positive and the equilibrium is guaranteed to be a nodal source. We say the system is structurally stable.

To repeat: the linear system with a nodal source (unstable equilibrium) is structurally stable.

Scenario 2. You have a known nonlinear system with a critical point at \( (x_0, y_0) \). You linearize the system and find that the linearized system has a nodal source with eigenvalues 1 and 7. In this case, structural stability tells us that the nonlinear system behaves like a nodal source close to the critical point.

That is, the approximation error changes some fine details of the system, but not the qualitative type of the system. We state this as a theorem

29.1 The open regions in the trace-determinant diagram are structurally stable

Theorem. The linearized system correctly classifies the critical point if the linear system is a spiral node, a nodal source or sink or a saddle.

It may not correctly classify a center, defective node, star node or non-isolated critical point.

That is, it is correct in open regions of the trace-determinant diagram and untrustworthy on the boundary lines.
The basic idea is that if we ‘jiggle’ the matrix it won’t move very far in the trace-determinant diagram, so the eigenvalues will be of the same type.

**Example 29.1.** Find the critical points for the system \( x' = y - x^2, \ y' = -x + y^2. \) Linearize at each critical point, and say whether the nonlinear system behaves like the linearized system near the point.

**answer:** Critical points: \( y - x^2 = 0 \Rightarrow y = x^2. \) Substitute this in the second equation to get \(-x + x^4 = 0 \Rightarrow x = 0, 1. \) Thus, there are two critical points \((0,0)\) and \((1,1)\).

Jacobian: \( J(x, y) = \begin{bmatrix} -2x & 1 \\ -1 & 2y \end{bmatrix}. \)

Linearizing:

\( J(1,1) = \begin{bmatrix} -2 & 1 \\ -1 & 2 \end{bmatrix} : \) characteristic equation: \( \lambda^2 - 3 = 0 \Rightarrow \lambda = \pm \sqrt{3} \Rightarrow \) linearized system has a saddle.

Since saddles are structurally stable the nonlinear system looks like a saddle at \((1,1)\).

\( J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} : \) eigenvalues = \( \pm i \) \Rightarrow a linearized center.

This is **not structurally stable**. The nonlinear system could be any one of a center, spiral out or spiral in. Using Matlab it appears that \((0,0)\) is in fact a center. (This can be proved.)
The following proof that the critical point is a center is only for those who are interested.

We can show the trajectories near \((0,0)\) are not spirals by exploiting the symmetry of the picture. First note, if \((x(t), y(t))\) is a solution then so is \((y(-t), x(-t))\). That is, the trajectory is symmetric in the line \(x = y\). This implies it can’t be a spiral. Since the only other choice is that the critical point \((0,0)\) is a center, the trajectories must be closed.

The following two examples show that a linearized center might also be a spiral sink or a spiral source in the nonlinear system.

**Example 29.2.** \(x' = y, \ y' = -x - y^3\).

**Example 29.3.** \(x' = y, \ y' = -x + y^3\).

In both examples the only critical point is \((0,0)\).

\[
J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow \text{linearized center. This is structurally unstable.}
\]

In Example 29.2 the critical point turns out to be a spiral sink. In Example 29.3 it is a spiral source.

Below are Matlab pictures. Because the \(y^3\) term causes the spiral to have a lot of turns we ‘improved’ the pictures by using the power 1.1 instead.

![Spiral in](image1.png) ![Spiral out](image2.png)

**29.1.1 Another proof, only for those who are interested.**

The proof that these are spirals in and out is based on Lyapunov’s second method using the potential function \(V(x,y) = x^2 + y^2\).

Consider the system \(x' = y, \ y' = -x - y^3\). If \((x(t), y(t))\) is a solution then \(\frac{dV}{dt} = 2x x' + 2y y' = -2y^4\). Since this is negative or 0 the potential \(V\) is decreasing along any trajectory of the system. That is, the trajectory must head towards the origin.

Thus, \((0,0)\) is an asymptotically stable critical point and its type must be a spiral sink.

Likewise for \(x' = y, \ y' = -x + y^3\); \(\frac{dV}{dt} = y^4 \geq 0 \Rightarrow V\) is increasing \(\Rightarrow\) trajectory heads away from origin \(\Rightarrow\) spiral source.