31 Applications to physics: mechanical systems

Nonlinear pendulum

\[ \theta'' + \frac{g}{l} \sin \theta = 0 \quad \text{or} \quad \theta'' + \omega^2 \sin \theta = 0. \] (Derivation given below.)

(For small \( \theta \) we approximate this by \( \theta'' + \omega^2 \theta = 0 \) – the linear pendulum.)

Using anti-elimination the nonlinear equation becomes the system

\[ \begin{align*}
x &= \theta & \Rightarrow x' &= y \\
y &= \theta' & \Rightarrow y' &= -\omega^2 \sin x
\end{align*} \]

Critical points: \((n\pi, 0)\) \( n = 0, \pm 1, \pm 2, \ldots \)

\[
J(x, y) = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos x & 0 \end{pmatrix} \Rightarrow \begin{cases} n \text{ even} & J = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \text{ center} \\
 n \text{ odd} & J = \begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix} \text{ saddle} \end{cases}
\]

Physically:

\( n \text{ even} \) (hanging down, stable) \( n \text{ odd} \) (Pointing up, unstable)

**Derivation of the pendulum equation**

There are many ways to derive this. We do it using rotational mechanics. Energy conservation is another good method.

Consider \( \theta \) to be positive in the counterclockwise direction. So in the picture, \( \theta'' < 0 \). We compute the torque about the pivot point.

Torque \( \vec{\tau} = \vec{l} \times \vec{F}_{\text{gravity}} \) has magnitude \( lmg \sin \theta \) and points straight down into the page.

We also know that \( |\vec{\tau}| = -ml^2 \theta'' \).

(The minus sign is because \( \theta'' < 0 \).)

\[ \Rightarrow \quad lmg \sin \theta = -ml^2 \theta'' \Rightarrow \theta'' = -\frac{g}{l} \sin \theta. \] QED
The labeled trajectories represent:
1. Round and round in a clockwise direction.
2. Just enough energy to stop at the unstable equilibrium.
3. Back and forth (like a, well, pendulum).
4. Like (2) in the opposite direction.
5. Like (1) in the opposite direction.

There are also the equilibria – not labeled on the plot;
(6) Stable (centers). *(unlabeled)*
(7) Unstable (saddles). *(unlabeled)*

**Note:**
The following useful trick allows us to solve for the trajectories exactly.
\[
\frac{dy}{dx} = \frac{y'}{x'} = -\frac{\omega^2 \sin x}{y}.
\]
This is separable and leads to \( y \, dy = -\omega^2 \sin x \, dx. \)
\[
\Rightarrow \frac{y^2}{2} = \omega^2 \cos x + E \Rightarrow \frac{y^2}{2} - \omega^2 \cos x = E.
\]
(We use \( E \) as the constant of integration to stand for energy.)
If \( m = 1 \) this is the usual 'total energy is constant' equation.

The motion of the pendulum depends on its total energy.
\( E > \omega^2 \Rightarrow \) round and round (trajectories (1), (5)).
\( -\omega^2 < E < \omega^2 \Rightarrow \) back and forth (trajectory (3)).
\( E = \omega^2 \Rightarrow \) unstable equilibrium (trajectories (2), (4), (7)).
\( E = -\omega^2 \Rightarrow \) stable equilibrium (trajectory (6)).
\( E < -\omega^2 \Rightarrow \) no trajectory.

**Damped nonlinear pendulum**
We add in damping: \( \theta'' + b\theta' + \omega^2 \sin \theta = 0. \)
\[ x = \theta, \ y = \theta' \Rightarrow \begin{cases} \ x' = \ y \\ \ y' = -\omega^2 \sin x - by. \end{cases} \]

As before, the critical points are at \((n\pi, 0)\) for any integer \(n\).

\[
J(x, y) = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos x & -b \end{pmatrix} \Rightarrow \begin{cases} \ n \text{ even} & J = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -b \end{pmatrix} \text{ sink} \\ \ n \text{ odd} & J = \begin{pmatrix} 0 & 1 \\ \omega^2 & -b \end{pmatrix} \text{ saddle} \end{cases}
\]

The type of linearized sink depends on the sign of the discriminant:

\[
b^2 - 4\omega^2 < 0 \Rightarrow \text{spiral sink} \\
b^2 - 4\omega^2 > 0 \Rightarrow \text{nodal sink}
\]

The pictures below show two underdamped nonlinear pendulums.

**Nonlinear Spring:**

\[ mx'' = -kx + \beta x^3: \begin{cases} \text{hard if} & \beta < 0 \text{ (cubic term adds to linear force)} \\ \text{soft if} & \beta > 0 \text{ (cubic term opposes linear force)} \end{cases} \]
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System: \[
\begin{align*}
x' &= y \\
m y' &= -kx + \beta x^3
\end{align*}
\]

Standard trick to get trajectories: \[
\frac{dy}{dx} = \frac{y'}{x'} = \frac{-kx + \beta x^3}{my}.
\]

Separable \(\Rightarrow \frac{my^2}{2} + \frac{kx^2}{2} - \frac{\beta x^4}{4} = E\) = constant.

If \(\beta < 0\) (hard spring) then both \(x\) and \(y\) are bounded and for fixed \(x\) there are at most two points on the trajectory.

\(\Rightarrow\) closed trajectories – we’ll find the period in a moment.

If \(\beta > 0\) (soft spring) then \(w = \frac{kx^2}{2} - \frac{\beta x^4}{4}\) has the graph given at right. Using the same graphical ideas as the proof that the Volterra predator-prey equation has closed trajectories this shows the phase plane for the soft spring is as shown below.

The labeled trajectories represent:
1. \(0 < E < \frac{k^2}{4\beta}\).
2. \(E > \frac{k^2}{4\beta}\).
3. \(E < \frac{k^2}{4\beta}\) (negative values included).

(In the hard spring \(E\) is always positive.)

**Period of a nonlinear spring** (See E& P §7.5 problems 17-20)

Consider the general equation \(x'' + \phi(x) = 0\). \(\Rightarrow\) system \[
\begin{align*}
x' &= y \\
y' &= -\phi(x)
\end{align*}
\]
Usual trick: \( \frac{dy}{dx} = -\frac{\phi(x)}{y} \)

Separate variables: \( \frac{1}{2}y^2 + \int_0^x \phi(u) \, du = E \) (a constant).

Let \( V(x) = \int_0^x \phi(u) \, du \) = potential function.

\[ \Rightarrow \frac{1}{2}y^2 + V(x) = E \] (defines trajectories implicitly).

Assume \( x(0) = x_0 \) and \( y(0) = 0 \) \( \Rightarrow E = V(x_0) = V_0 \).

\[ \Rightarrow y = \pm \sqrt{2(V_0 - V(x))} = \frac{dx}{dt}. \]

(Which of the \( \pm \) we need depends on which quadrant the phase-plane trajectory is in.)

For the spring the equation \( \frac{1}{2}y^2 + \frac{kx^2}{2} - \frac{\beta x^4}{4} = E \) gives a symmetric trajectory.

The plot at right shows 1/4 of a closed trajectory starting at \((x_0, 0)\). (The closed trajectories are the ones that start on the \(x\)-axis and have \(y' < 0\).) \( \Rightarrow \frac{dx}{dt} = -\sqrt{2(V_0 - V(x))}. \)

Separate variables: \( dt = -\frac{dx}{\sqrt{\frac{kx_0^2}{2} - \frac{\beta x^4}{4} - \frac{kx_0^2}{2} + \frac{\beta x_4}{4}}}. \)

\[ \Rightarrow \text{period } T = 4 \int_0^{x_0} \frac{dx}{\sqrt{\frac{kx_0^2}{2} - \frac{\beta x_4}{4} - \frac{kx_0^2}{2} + \frac{\beta x_4}{4}}}. \]

To understand what this means see the next page.

Let \( T_0 = \frac{2\pi}{\sqrt{k}} \) (the linearized period), \( \epsilon = \frac{\beta}{k} x_0^2 \), \( \mu = -\frac{1}{2} \frac{\epsilon}{1 - \epsilon} \).

Substitute \( x = x_0 \cos \phi \Rightarrow \)
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\[ T = \frac{4}{\sqrt{2}} \int_{0}^{\pi/2} \frac{x_0 \sin \phi \, d\phi}{\left( \frac{kx_0^2}{2} (1 - \cos^2 \phi) - \frac{\beta x_0^4}{4} (1 - \cos^4 \phi) \right)^{1/2}} \]

\[ = \frac{4}{\sqrt{2}} \int_{0}^{\pi/2} \frac{\sqrt{2}}{\sqrt{2}} \frac{d\phi}{(\frac{k}{2} - \frac{\beta}{4} (1 + \cos^2 \phi))^{1/2}} \]

\[ = \frac{4}{\sqrt{k}} \int_{0}^{\pi/2} \frac{1}{\sqrt{1 - \epsilon (1 - \mu \sin^2 \phi)^{1/2}}} \frac{d\phi}{\sqrt{1 - \epsilon (1 - \mu \sin^2 \phi)^{1/2}}} \]

\[ = \frac{2}{\sqrt{1 - \epsilon}} \int_{0}^{\pi/2} \frac{1}{\sqrt{1 - \epsilon (1 - \mu \sin^2 \phi)^{1/2}}} \frac{d\phi}{\sqrt{1 - \epsilon (1 - \mu \sin^2 \phi)^{1/2}}} \]

Note, \( \beta = 0 \Rightarrow \epsilon = \mu = 0 \Rightarrow T = T_0. \)

If \( \mu \) is small then

\[ \frac{1}{(1 - \mu \sin^2 \phi)^2} \approx 1 + \frac{1}{2} \mu \sin^2 \phi \Rightarrow T \approx \frac{2T_0}{\sqrt{1 - \epsilon}} \left( \frac{\pi}{2} + \mu \frac{\pi}{8} \right) = \frac{T_0}{\sqrt{1 - \epsilon}} (1 + \frac{\mu}{4}). \]

**Damped nonlinear spring:** \( mx'' = -kx + \beta x^3 - cx' \)

\[ \Rightarrow \begin{cases} 
  x' = y \\
  y' = -kx - cy + \beta x^3 
\end{cases} \]

Hard spring \((\beta < 0)\): One critical point at \((0, 0)\)

\[ J(0, 0) = \begin{pmatrix} 0 & 1 \\ -k/m & -c/m \end{pmatrix} \Rightarrow \lambda = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}. \]

\( \Rightarrow \) overdamped = spiral sink; overdamped = nodal sink; critically damped = defective sink.

Soft spring \((\beta > 0)\): Critical points: \((0, 0), (\pm \sqrt{k/\beta}, 0)\).

\((0, 0)\) – sink (spiral, nodal or defective); \((\pm \sqrt{k/\beta}, 0)\) – saddles.