6 Operators, inhomogeneous DEs, exponential response formula

6.1 Goals

1. Be able to define linear differential operators.
2. Be able to define polynomial differential operators and use them to express linear constant coefficient differential equations.
3. Be able to use the Exponential Response Formula to find particular solutions to polynomial differential equations with exponential or sinusoidal input.
4. Be able to derive the Sinusoidal Response Formula.
5. Be able to use the Sinusoidal Response Formula to solve polynomial differential equations with sinusoidal input.
6. Be able to build models of damped harmonic oscillators with input.

6.2 Linear Differential Equations

Linear $n$th-order differential equations have the form

\[ p_0(t)y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0 \]  \hspace{1cm} (H)

\[ p_0(t)y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = f(t) \]  \hspace{1cm} (I)

As usual, we call (H) homogeneous and (I) inhomogeneous.

Also as usual, if the coefficients are all constant then we have a constant coefficient linear differential equation.

\[ a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = 0 \]  \hspace{1cm} (H)

\[ a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = f(t) \]  \hspace{1cm} (I)

In Topic 5 we learned about the characteristic equation

\[ a_0r^n + a_1r^{n-1} + \cdots + a_n = 0 \]

It will be useful to give a name to the polynomial

\[ P(r) = a_0r^n + a_1r^{n-1} + \cdots + a_n. \]

We will call it the characteristic polynomial. That is, the characteristic equation can be written $P(r) = 0$. 


6 OPERATORS, INHOMOGENEOUS DES, EXPONENTIAL RESPONSE FORMULA

6.3 Operators

A function is a rule that takes a number as input and returns another number as output.

Example 6.1. (Examples of functions.)
1. \( f(t) = t^2 \). If \( t = 2 \) is the input then \( f(t) = 4 \) is the output.
2. The identity function is \( f(t) = t \).
3. The zero function is \( f(t) = 0 \).

An operator is similar to a function except that it takes as input a function and returns another function as output. We will often use upper case letters like \( T \) or \( L \) to denote operators. If \( x \) is a function when \( T \) acts on it we will write \( T(x) \) or \( Tx \).

We will read this as “\( T \) of \( x \)” or “\( T \) applied to \( x \)” or “\( T \) acting on \( x \).” A few examples will make this clear.

Example 6.2. The differentiation operator is \( D = \frac{d}{dt} \). This takes any function as input and returns its derivative as output. For example,
(i) If \( x(t) = t^3 \) then \( D(x) = 3t^2 \). We also write \( Dx = 3t^2 \).
(ii) If \( y(t) = e^{4t} \) then \( Dy = 4e^{4t} \).
(iii) \( D(t^3 + 2t^2 + 5t + 7) = 3t^2 + 4t + 5 \).
(iv) In general, \( Dx = x' \).

Example 6.3. The second derivative operator is \( D^2 = \frac{d^2}{dt^2} \). For example:
(i) \( D^2(e^{4t}) = 4^2e^{4t} \).
In this example we used \( D^2 \) to mean first apply \( D \) to the function and then apply it again. Writing this out in more detail we get
\[
D^2(e^{4t}) = D(D(e^{4t})) = D(4e^{4t}) = 4^2e^{4t}.
\]
(ii) In general, \( D^2x = x'' \). Likewise, \( D^3 = x''' \).

Example 6.4. The identity operator \( I \) takes any function as input and returns the same function as output. For example:
(i) \( I(x) = x \).
(ii) \( I(t^2 + 3t + 2) = t^2 + 3t + 2 \).

Example 6.5. We can combine these operators. For example we can let
\[
T = D^2 + 8D + 7I.
\]
To understand what this operator does we have to apply it to a function and see what happens. If we apply \( T \) to \( x \) we get
\[
Tx = (D^2 + 8D + 7I)x = x'' + 8x' + 7x.
\]
Comment. Writing $7I$ is the formal way to express this. In class and occasionally in writing I’ll write $(D^2 + 8D + 7)x$. This is not really grammatical, but it’s fairly common usage.

Example 6.6. The zero operator takes any function as input and returns the zero function as output. There is no standard notation for this function, let’s call it $Z$. For example:

(i) $Z(x) = 0$.
(ii) $Z(t^2 + 3t + 2) = 0$.

For obvious reasons we call $D$, $D^2$, $D^3$, . . . differential operators.

6.4 Polynomial differential operators

Consider the polynomial $P(r) = r^2 + 8r + 7$. If we replace $r$ by $D$ we have $P(D) = D^2 + 8D + 7$. We will call $P(D)$ a polynomial differential operator. We can use it to simplify writing down DEs and to help with algebraic manipulations.

Example 6.7. Consider the constant coefficient differential equation

$$x'' + 8x' + 7x = 0.$$  

This has characteristic polynomial $P(r) = r^2 + 8r + 7$. Using this we can rewrite the DE in polynomial notation as

$$(D^2 + 8D + 7I)x = 0$$  

or even more simply

$$P(D)x = 0.$$  

One great thing about polynomial operators is how simply we can express constant coefficient differential equations using them. We can rewrite (H) and (I) above as

$$P(D) = 0 \quad \text{(H)}$$
$$P(D) = f(t) \quad \text{(I)}$$

where $P(D) = D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_nI$.

6.5 Linearity/superposition for polynomial differential operators

The superposition principle was awkward to state and prove because it was phrased in terms of equations. Linearity is equivalent to superposition, but easier to discuss because we phrase it in terms of operators.

Important definition. An operator $T$ is called a linear operator if for any functions $x_1$, $x_2$ and any constants $c_1$, $c_2$ we have

$$T(c_1x_1 + c_2x_2) = c_1Tx_1 + c_2Tx_2.$$  

Claim. Show that the differential operator $D$ is linear.
Proof. This is easy to check directly from the definition of linearity:

\[ D(c_1 x_1 + c_2 x_2) = (c_1 x_1 + c_2 x_2)' = c_1 x_1' + c_2 x_2' = c_1 D x_1 + c_2 D x_2 \]

Looking at the first and last terms in this string of equalities we see that Equation 1 holds for the operator \( D \).

Similarly we can show that the operators \( D^2 \), \( D^3 \) are linear. Likewise for any polynomial the operator \( P(D) \) is linear.

Example 6.8. Show directly from the definition that \( P(D) = D^2 + 8D + 7I \) is linear.

answer: We use the same argument as in the proof of the claim just above:

\[ P(D)(c_1 x_1 + c_2 x_2) = (c_1 x_1 + c_2 x_2)'' + 8(c_1 x_1 + c_2 x_2)' + 7(c_1 x_1 + c_2 x_2) \]
\[ = c_1 (x_1'' + 8x_1' + 7x_1) + c_2 (x_2'' + 8x_2' + 7x_2) \]
\[ = c_1 P(D)x_1 + c_2 P(D)x_2 \]

I hope the examples have convinced you that the linearity of an operator is easy to verify. You might also have noticed how similar the arguments felt to those showing the superposition principle. For completeness we state and show that the two are equivalent.

Equivalence of linearity and the superposition principle. Suppose \( T \) is an operator. Then \( T \) is linear if and only if the equation \( Tx = q(t) \) satisfies the superposition principle.

Proof. This is really just a matter of unwinding the definitions. Suppose \( Tx_1 = q_1 \) and \( Tx_2 = q_2 \). If the superposition principle holds then

\[ T(c_1 x_1 + c_2 x_2) = c_1 q_1 + c_2 q_2 = c_1 T x_1 + c_2 T x_2. \]

Likewise, if \( T \) is linear then

\[ T(c_1 x_1 + c_2 x_2) = c_1 T x_1 + c_2 T x_2 = c_1 q_1 + c_2 q_2. \]

The above two equations are equivalent. Therefore, so are linearity and superposition.

6.6 The algebra of \( P(D) \) applied to exponentials

For this section \( P(D) \) will be a polynomial differential operator and \( a \) will be a constant. Here are two easy and useful rules concerning \( P(D) \) and \( e^{at} \). We will use them immediately to show why we have factors of \( t \) in the solutions to DEs with repeated roots.

6.6.1 Substitution rule

Substitution rule. \( P(D)e^{at} = P(a)e^{at} \). This is called the substitution rule because we just substitute \( a \) for \( D \).

‘Proof’ by example. We show the rule holds for \( P(r) = r^2 + 8r + 7 \):

\[ P(D)e^{at} = (e^{at})'' + 8(e^{at})' + 7e^{at} = (a^2 + 8a + 7)e^{at} = P(a)e^{at}. \]
6.6.2 Exponential shift rule

We will call $P(D + aI)$ a shift of $P(D)$ by $a$. For example, if $P(D) = D^2 + 2D + I$ then

$$P(D - I) = (D - I)^2 + 2(D - I) + I = D^2 - 2D + I + 2D - 2I + I = D^2.$$ 

**Exponential shift rule for $D$.** For any function $u(t)$,

$$D(e^{at}u(t)) = e^{at}(D + aI)u(t).$$

**Proof.** The derivation of this is just the product rule for differentiation:

$$D(e^{at}u(t)) = ae^{at}u(t) + e^{at}u'(t) = e^{at}(au(t) + u'(t)) = e^{at}(D + aI)u(t).$$

**Exponential shift rule for $D^2$.** For any function $u(t)$,

$$D^2(e^{at}u(t)) = e^{at}(D + aI)^2u(t).$$

A similar statement holds for $D^3$, $D^4$, . . .

**Proof.** To derive this for $D^2$ we just use the rule for $D$ twice. Higher powers are similar.

Now it is clear (by linearity!) that the rule applies to any $P(D)$:

**Exponential shift rule for $P(D)$.** For any function $u(t)$,

$$P(D)(e^{at}u(t)) = e^{at}P(D + aI)u(t).$$

6.6.3 Repeated roots

We are now in a position to explain the rule for solutions with real roots. Recall:

**Rule for repeated roots.** If the characteristic equation $P(r)$ has a repeated root $r_1$ then both $x_1(t) = e^{r_1t}$ and $x_2(t) = te^{r_1t}$ are solutions to the homogeneous DE $P(D)x = 0$.

‘Proof’ by example. Use the exponential shift rule to show the the equation $x'' - 6x' + 9 = 0$ has general solution $x(t) = c_1e^{3t} + c_2te^{3t}$.

**Answer:** First we rewrite this equation in terms of $P(D)$. The characteristic polynomial is

$$P(r) = r^2 - 6r + 9 = (r - 3)^2.$$ 

So $P(D) = (D - 3)^2$ and the differential equation is $P(D)x = 0$.

We know $P(r)$ has repeated roots $r = 3, 3$. So, $x(t) = c_1e^{3t}$ is a solution. Let’s vary the parameters to look for other solutions, i.e. let’s try $x(t) = e^{3t}u(t)$. We substitute this into the equation and apply the shift rule:

$$P(D)x = 0$$

$$= P(D)(e^{3t}u)$$

$$= e^{3t}P(D + 3I)u$$

$$= e^{3t}(D + 3I - 3I)^2u$$

$$= e^{3t}D^2u.$$
Thus we have the equation $D^2 u = 0$, i.e. $u''(t) = 0$. This is an 18.01 problem and the solution is $u(t) = c_1 + c_2 t$. Putting this back into $x(t)$ we have found

$$x(t) = e^{3t}(c_1 + c_2 t),$$

which is exactly what the rule for repeated roots rule said we would find.

### 6.6.4 Complexification example

**Example 6.9.** Compute $D^3(e^x \sin(x))$.

**answer:** Using complexification $e^x \sin(x) = \text{Im}(e^{x+ix})$. So $D^3(e^x \sin(x)) = \text{Im}(D^3(e^{x+ix}))$.

Computing this we have

$$D^3(e^{x+ix}) = (1 + i)^3 e^{x+ix} = 2^{3/2} e^{3\pi/4} e^{ix} = 2^{3/2} e^{i(x+3\pi/4)}$$

Taking the imaginary part we have

$$D^3(e^x \sin(x)) = \text{Im}(D^3(e^{x+ix})) = 2^{3/2} e^{x} \sin(x + 3\pi/4).$$

### 6.7 Exponential Response Formula

This is one of the key formulas we will use throughout the rest of ES.1803, especially in the form that deals with cosines and sines.

**Exponential Response Formula.** Let $P(D)$ be a polynomial differential operator. The inhomogeneous constant coefficient linear DE $P(D)y = e^{at}$ has a particular solution

$$y_p(t) = \begin{cases} 
  e^{at}/P(a) & \text{provided } P(a) \neq 0 \\
  te^{at}/P'(a) & \text{if } P(a) = 0 \text{ and } P'(a) \neq 0 \\
  t^2e^{at}/P''(a) & \text{if } P(a) = P'(a) = 0 \text{ and } P''(a) \neq 0 \\
  \ldots & \ldots 
\end{cases}$$

**Simple proof:** The substitution rule says

$$P(D)e^{at} = P(a)e^{at}. \quad (2)$$

If $P(a) \neq 0$ then dividing 2 by $P(a)$ proves the theorem in this case.

If $P(a) = 0$ we differentiate 2 with respect to $a$. This gives

$$P(D)(te^{at}) = P'(a)e^{at} + P(a)te^{at}.$$ 

Since $P(a) = 0$, the second term on the left is 0 and we have $P(D)(te^{at}) = P'(a)e^{at}$. Dividing by $P'(a)$ proves the theorem in the case $P(a) = 0$ and $P'(a) \neq 0$.

We can continue in this manner if $P(a) = P'(a) = 0$ etc.
Notes: 1. The Exponential Response Formula (ERF) is also called the Exponential Input Theorem or EIT.
2. We will call the cases where $P(a) = 0$ the Extended Exponential Response Formula.
3. You will need to know how to use the Extended ERF. You will not be asked to know the proof –although doing so is certainly good for you.

Example 6.10. Let $P(D) = D^2 + 4D + 5I$ ($P(r)$ has roots $-2 \pm i$).

(a) Find a solution to $P(D)x = e^{-t}$.

(b) Find a solution to $P(D)x = \cos(2t)$.

(c) Find a solution to $P(D)x = e^{-t}\cos(2t)$,

**answer:** (a) The question only asks for one solution not all of them. The equation has exponential input, so, since $P(-1) = 2$ the exponential response formula says $x_p = e^{-t}/2$ is a solution.

(b) (Long form of the solution.) We first complexify the equation by replacing $\cos(2t)$ by the complex exponential $e^{2it}$:

Introduce $y(t)$ such that $P(D)y = \sin(2t)$. Then by linearity $z(t) = x(t) + iy(t)$ satisfies the DE

$$P(D)z = \cos(2t) + i\sin(2t) = e^{2it}$$

and $x = \text{Re}(z)$.  

In preparation for using the Exponential Response Formula we first compute $P(2i)$ in polar form. We have $P(2i) = 1 + 8i$. Therefore

$|P(2i)| = |1+8i| = \sqrt{65}$ and $\phi = \text{Arg}(P(2i)) = \text{Arg}(1 + 8i) = \tan^{-1}(8)$ in quadrant 1.

Thus we have $P(2i) = \sqrt{65}e^{i\phi}$. The ERF gives us complex-valued solution to 3:

$$z_p(t) = \frac{e^{2it}}{P(2i)} = \frac{e^{2it}}{\sqrt{65}e^{i\phi}} = \frac{e^{(2t-\phi)}}{\sqrt{65}}.$$

All that’s left is to take the real part to get a solution to the original DE:

$$x_p(t) = \text{Re}(z_p(t)) = \frac{\cos(2t - \phi)}{\sqrt{65}}.$$

**Note:** The general form of the solution is

$$z_p = \frac{e^{2it}}{P(2i)}$$

and $x_p = \frac{1}{|P(2i)|} \cos(2t - \phi)$, where $\phi = \text{Arg}(P(2i))$.

Make sure you understand this statement thoroughly. We will use it repeatedly.

(c) (Short form of solution.) Complexify the DE:

$$P(D)z = e^{-t}e^{2it} = e^{(-1+2i)t},$$

where $x = \text{Re}(z)$.

Side work: $P(-1 + 2i) = -2 + 4i = 2\sqrt{5}e^{i\phi}$, where $\phi = \text{Arg}(-2 + 4i) = \tan^{-1}(-2)$, $\phi$ in second quadrant.
ERF: $z_p(t) = \frac{e^{(-1+2i)t}}{P(-1+2i)} = \frac{e^{(-1+2i)t}}{-2+4i} = \frac{e^{-t}e^{2it}}{2\sqrt{5}e^{i\phi}}$.

Algebra: $z_p(t) = \frac{e^{-t}e^{2it}}{2\sqrt{5}e^{i\phi}} = \frac{e^{-t}e^{2it}}{2\sqrt{5}e^{i\phi}} = e^{-t}\sqrt{5}e^{i\phi}$. Thus, $x_p = \text{Re}(z_p) = \frac{e^{-t}}{2\sqrt{5}}\cos(2t - \phi)$.

**Example 6.11.** With the same $P(D)$ as in the previous example, find a solution to $P(D)x = e^{-2t}\cos(t)$

**Answer:** Complexify: $P(D)z = e^{-2t}e^{it} = e^{(-2+i)t}$ where $x = \text{Re}(z)$.

Side work: $P(-2 + i) = 0$, so we’ll need $P'(-2 + i)$.

$P'(r) = 2r + 4$, So $P'(-2 + i) = 2i = 2ei\pi/2$.

ERF: $z_p(t) = \frac{te^{(-2+i)t}}{P'(-2+i)} = \frac{te^{(-2+i)t}}{2e^{i\pi/2}} = \frac{te^{-2t}e^{(t-\pi/2)}}{2}$

Real part: $x_p(t) = \text{Re}(z(t)) = \frac{te^{-2t}}{2} \cos(t - \pi/2)$.

**Get good at this, we will do it a lot**

### 6.8 The Sinusoidal Response Formula

In the examples above we saw a pattern when the input was sinusoidal. We use it so often that we will codify the result as the Sinusoidal Response Formula.

**Sinusoidal Response Formula.** Consider the polynomial differential equation $P(D)x = \cos(\omega t)$

If $P(i\omega) \neq 0$ then the DE has a particular solution $x_p(t) = \frac{1}{|P(i\omega)|} \cos(\omega t - \phi)$, where $\phi = \text{Arg}(P(i\omega))$.

If $P(i\omega) = 0$ we have the **Extended SRF**. For example, if $P(i\omega) = 0$ and $P'(i\omega) \neq 0$ then the DE has a particular solution $x_p(t) = \frac{t\cos(\omega t - \phi)}{|P'(i\omega)|}$, where $\phi = \text{Arg}(P'(i\omega))$.

**Proof.** To prove the extended SRF we just follow the steps from the examples above.

1. Complexify: $P(D)z = e^{i\omega t}$, where $x = \text{Re}(z)$.
2. Write $P'(i\omega)$ in polar coordinates: $P'(i\omega) = |P'(i\omega)|e^{i\phi}$, where $\phi = \text{Arg}(P'(i\omega))$.
3. Use the extended ERF: $z_p = \frac{te^{i\omega t}}{P'(i\omega)} = \frac{te^{i(\omega t - \phi)}}{|P'(i\omega)|}$.
4. Find the real part of $z_p$: $x_p(t) = \text{Re}(z_p(t)) = \text{Re} \left( \frac{te^{i(\omega t - \phi)}}{|P(i\omega)|} \right) = \frac{t\cos(\omega t - \phi)}{|P(i\omega)|}$.

Remember: if in doubt when using the extended SRF you can always derive it using complexification and the extended ERF.
6.9 Physical models

In this section we will look at three versions of the driven spring-mass-dashpot. These have analogies, which we won’t show here, in RLC circuits. These are relatively simple second order systems, but the same reasoning can be used to develop models for more complicated systems.

In all three examples we assume linear damping with damping constant $b$. That is, if the damper is moving with velocity $v$ through the dashpot, then the force of the dashpot on the damper is $-bv$. This is a reasonable model if the dashpot is filled with a viscous oil.

Example 6.12. Driving through the mass. In this version there is a spring-mass-dashpot which is driven by a variable force applied to the mass as shown. The position of the mass is $x(t)$, with $x = 0$ being the equilibrium position, i.e. the position where the spring is relaxed. Our goal in this example is to build a DE modeling $x(t)$.

![Diagram](image)

To model this we consider all the forces on the mass use Newton’s law. The spring is stretched by $x$, so it exerts a restoring force: $-kx$. The velocity of the damper through the dashpot is $\dot{x}$, so it exerts a resisting force: $-b\dot{x}$. Thus Newton’s law gives

$$m\ddot{x} = -kx - b\dot{x} + F(t) \iff m\ddot{x} + b\dot{x} + kx = F(t).$$

Example 6.13. Driving through the spring. In this version the spring-mass-dashpot is driven by a mechanism that positions the end of the spring at $y(t)$ as shown. As before $x(t)$ is position of the mass. We calibrate $x$ and $y$ so that $x = 0, y = 0$ is an equilibrium position of the system.

![Diagram](image)

To model this we must consider all the forces on the mass. At time $t$ the spring is stretched an amount $x(t) - y(t)$, so the spring force is $-k(x - y)$. Likewise the velocity of the damper through the dashpot is $\dot{x}$ so the damping force is $-b\dot{x}$. Thus,

$$m\ddot{x} = -k(x - y) - b\dot{x} \iff m\ddot{x} + b\dot{x} + kx = ky.$$

Example 6.14. Driving through the dashpot. In this version the spring-mass-dashpot is
driven by a mechanism that positions the end of the dashpot at $y(t)$ as shown. Again, $x(t)$ is position of the mass and $x = 0, y = 0$ is an equilibrium position of the system.

More briefly than the previous examples:

spring force: $-kx$

damping force: $-b(\dot{x} - \dot{y})$.

Model: $m\ddot{x} = -kx - b(\dot{x} - \dot{y}) \Leftrightarrow m\ddot{x} + b\dot{x} + kx = b\dot{y}$. 