Linear Algebra and Systems of DE’s

Problem 1
Consider the system \( x' = -3x + 2y, \quad y' = -x - y. \)
Find the solution \( x(t), y(t) \) satisfying the IC’s \( x(0) = 0, y(0) = 1 \) using the eigenvalue/eigenvector method.

Problem 2
Consider the system \( x' = 5x - 6z, \quad y' = 2x - y - 2z, \quad z' = 4x - 2y - 4z. \)
(a) Rewrite this system of DE’s in matrix form \( x' = Ax. \)
(b) Given that the eigenvalues of the matrix \( A \) are 0, -1 and 1, write down the form of the three normal modes \( x_1, x_2, x_3 \) for this system without solving the system explicitly, i.e. just give the eigenvectors names, but don’t find them.
(c) What is the long-run behavior (i.e. as \( t \to \infty \)) of the general solution to this system of DE’s? Justify your answer (briefly).
(d) Find the eigenvectors and write the explicit solution to the system.
(e) Diagonalize the coefficient matrix.
(f) Decouple the system.

Problem 3
Suppose the two compartment system with flow rates and volumes (in some suitable units) as shown has constant concentrations of solute \( \alpha \) and \( \beta \) in the inflows into tank 1 and 2 respectively.

(a) Give the system of DE’s modeling the amounts of solute \( x(t), y(t) \) in tanks 1 and 2. Find a particular solution \( x_p \) to this inhomogeneous DE which is constant. (Give your answer in terms of \( \alpha \) and \( \beta \).)
(b) Show that the solution \( x_p \) found in part (a) is the “steady-state” solution for this system, in the sense that all solutions to this system approach this particular solution \( x_p \) as \( t \) goes to \( \infty \). (Note: this can be done without lots of calculation.)

Problem 4
Let \( A = \begin{bmatrix} 2 & 12 \\ 3 & 2 \end{bmatrix} \)
(a) What are the eigenvalues of $A$?
(b) For each eigenvalue, find a nonzero eigenvector.

**Problem 5**
Suppose that the matrix $B$ has eigenvalues 1 and 2, with eigenvectors

\[
\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\]
respectively.

What is the solution to $x' = Bx$ with $x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$?

**Problem 6**
Let $A = \begin{bmatrix} a & -2 \\ 2 & 1 \end{bmatrix}$, and consider the homogeneous linear system $x' = Ax$. Determine all values of $a$ (if any) for which the system is stable.

**Problem 7**
Convert the following to a system of DE’s and use matrix methods to find a solution for $x(t)$

\[ \dot{x} + 2\dot{x} + 2x = 0. \]

**Problem 8**
Let $A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

(a) Find the reduced echelon form of $A$.
(b) What is the rank of $A$?
(c) Find a basis for the null space of $A$.
(d) Find a basis for the column space of $A$.

(e) Find a matrix with the same reduced echelon form but such that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ are in its column space.

**Problem 9**
Let $A = \begin{bmatrix} a & 1 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

(a) What are the eigenvalues of $A$?
(b) For what value (or values) of $a$ is $A$ singular (non-invertible)?
(c) What is the minimum rank of $A$ (as $a$ varies)? What’s the maximum?
Problem 10
Suppose that \( A = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} S^{-1} \). Where \( S \) is an invertible 3 \times 3 matrix.

(a) What are the eigenvalues of \( A \)?

(b) Express \( A^2 \), \( A^{-1} \) in terms of \( S \).

(c) For the system \( \mathbf{x}' = A \mathbf{x} \), is the equilibrium at the origin stable, unstable, or neither?

(d) What would I need to know about \( S \) in order to write down the most rapidly growing exponential solution to \( \mathbf{x}' = A \mathbf{x} \)?

Problem 11

(a) Find an orthogonal matrix \((S^T S = I)\) and a diagonal matrix \( \Lambda \) such that \( \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = S \Lambda S^{-1} \). Don’t worry about the term orthogonal, we didn’t do it in class. The problem just asks you to diagonalize the matrix. Notice that the eigenvectors are perpendicular to each other.

(b) Decouple the equation \( \mathbf{x}' = A \mathbf{x} \), with \( A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \); that is, find coordinates \( u_1 = ax_1 + bx_2, u_2 = cx_1 + dx_2 \), such that the equation is equivalent to \( u_1' = \lambda_1 u_1, u_2' = \lambda_2 u_2 \).

First Order DE Questions

Problem 1
For the DE \( \frac{dy}{dx} = x^2 - y \):

(a) Sketch the direction field for this DE, using (light or dotted) isoclines for the slopes \(-2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1\).

(b) Sketch in a ‘guess’ for the integral curve which passes through the point \( (0, 2) \) using only the direction field information.

(c) If Euler’s method with step-size \( h = 0.05 \) was used to approximate \( y(0.1) \) for the integral curve of part (b), would the approximation come out too high or too low? Explain.

(d) Solve the DE with the IC \( y(0) = 2 \) to get an exact solution. Then compute the Euler approximation to \( y(0.1) \) using step-size \( h = 0.05 \), and verify the prediction made in part(c).

Problem 2
We have a function \( f(x, y) \) which we know is continuous. We also know the following
\[ f(x, y) = -1 \text{ along the curve } y = e^{-x}. \]
\[ f(x, y) = 1 \text{ along the curve } y = -e^{-x}. \]

From this we can deduce the behavior for large \( x \) of certain solutions of the DE \( y' = f(x, y) \). Which solutions, and what can we say about them?

(You might begin by sketching the two known isoclines.)

Problem 3
Suppose that a population of variable size (in some suitable units) \( P(t) \) follows the growth
law $\frac{dP}{dt} = -P^3 + 12P^2 - 36P + r$, where $r$ is a constant replenishment rate. Without solving the DE explicitly:

(a) Let $r = 0$. Find all critical points and classify each according to its stability type using a phase-line diagram. Sketch some representative integral curves.

(b) What is the smallest value of $r$ such that the population always stabilizes at a size greater than 4, no matter what the size of the initial population?

(c) Sketch the $P$ vs. $r$ bifurcation diagram.

Remember to look over your notes and problem sets. The practice problems do not cover absolutely everything.