18.03 Problem Set 1, Fall 2017
(due in class on Thursday, Sep. 14)

Part I (10 points)

EP = Edwards & Penney; SN = Supplementary Notes (all have solutions)
Problems marked 'Others' are not to be handed in.

Topic 1 (W, Sep. 6) Introduction to DEs; modeling; separable equations.
Read: Class notes topic 1, SN §D.
Hand in: Topic 1 part I problems: 1-2 (posted with pset); SN 1A/3a, 4a, 5b; 1D/1a.

Read: Class notes topic 2; SN §IR1-4.
Hand in: Topic 2 part I problems 1-2: (posted with pset); SN 1B 6; 1D/5.

Part II (79 points)

Directions: Try each problem alone for 20 minutes. If, after this, you collaborate, you
must write up your solutions independently.

Problem 1 (Topic 1) (15: 5,5,5)
Suppose that a ship’s engines provide a constant thrust force \( T \) and that the resistance force
of the water is proportional, with constant \( a \), to the square of its velocity.

(a) Write down the DE for the ship’s velocity \( v(t) \) as a function of time. Use the ‘rest’
initial condition \( v(0) = 0 \) (i.e. the ship starts from rest at time zero) Use \( m \) for the mass of
the ship.

(b) Without solving the DE, find the terminal velocity \( v_\infty = \lim_{t \to \infty} v(t) \) of the ship.
Show that your formula for \( v_\infty \) has the correct units (length/time).

(c) Solve the DE with rest IC, sketch a graph of the solution \( v = v(t) \), and also verify the
answer you got in part(b).

Problem 2 (Topic 3) (10: 6,4)
(a) This problem is warmup for problems 3 and 4. It is about matching initial and final
conditions at transition points of the input.
Solve the DE \( x' + kx = f(t) \), \( x(0) = 0 \) where the input \( f(t) \) is given by
\[ f(t) = \begin{cases} 
1 & \text{for } 0 \leq t < 0.5 \\
0 & \text{for } 0.5 \leq t < 1 \\
1 & \text{for } 1 \leq t 
\end{cases} \]

Give the solution \( x(t) \) in cases format (e.g. the function \( f(t) \) above is given in cases format).

Use the ‘trick’ \( e^{-k(t-a)} \), as in Circuit Example in topic 3 notes. For example, in the interval \( 0.5 \leq t < 1 \) the solution is \( x(t) = C_1 e^{-k(t-0.5)} \), where \( C_1 = x(0.5) \) matches the value of \( x(t) \) at the end of the interval \( 0 \leq t < 0.5 \).

(b) For \( k = 1 \): graph the solution for part (a). If this DE is modeling a mixing-tank situation, describe in words the behavior of the level of salt in the tank that this graph is showing, and explain how the behavior of the response relates to the input salt rate \( f(t) \).

**Problem 3**  
(Topic 3)  
(20: 6,5,9)

A population of lemmings, crazed by global warming, has been flinging themselves into the sea at a rate faster than they can reproduce. As a result the deathrate of the lemmings is now greater than the birthrate, so the population is in decline. Studies show that if nothing is done then the lemming population is halved every 2 years.

(a) Let \( x \) be the number of lemmings, \( t \) the time in years, \( k > 0 \) the decay rate of the lemming population.

i. Modeling this with continuous variables show that the DE for \( x \) as a function of \( t \) is \( x' + kx = 0 \). (We’re just looking for a one line answer.)

ii. Give the value of \( k \), including units.

iii. Assume that at time \( t_0 \) the population is \( x_0 \). Give the solution to the DE of part this that satisfies this initial condition. (This gives the behavior of the system, i.e. the number of lemmings, when there is no input.)

(b) Alarmed by the potential loss of tourist business if the big annual lemming run should disappear, an importation/stocking program has been introduced by the Greenland Chamber of Commerce. For 6 months of the year (Jan. 1 to June 30) they import Canadian lemmings at a constant rate of \( r \) lemmings per year. For the other 6 months (July 1 to Dec. 31) the lemmings are left to their own devices.

i. Show that \( x' + kx = r \) is the DE that models the population when restocking is occurring. (Again, one line will suffice.)

ii. Assume that at time \( t_0 \) the population is \( x_0 \). Give the solution to this DE that satisfies this initial condition. (This is gives the response of the system, i.e. the number of lemmings, when there is a constant input \( r \) lemmings/year.)

(c) Assume on Jan. 1, 2016 the population was 3000. Also, assume the stocking rate \( r \) is 1680 lemmings per year. Alternately using your answers to parts (a) and (b) find the lemming population on Jan. 1, 2018. (For the purposes of this problem you should take the period Jan. 1 to July 1 to be exactly .5 years and \( k = .35 \).) Be sure to use consistent units.

**Problem 4**  
(Topic 3)  
(20: 5,5,5,5)

We continue with the Lemmings model in problem 3.

(a) In general, a system with exponential decay and input function \( f(t) \) can be written as \( x' + kx = f(t) \). For the system described in problem (3), write down the input \( f(t) \) using cases format. Do this over enough cycles of no stocking/restocking that the pattern
is clear.

(b) Write down the solution to the DE in part (a): Give the answer in cases format for the time interval $0 < t < 2$.

(Hint: you have already done all the work in problem (3). This problem just asks you to organize it into a solution.

(c) Using the answer to part (b), draw a rough plot of the system response to this on again-off again input for $0 \leq t \leq 2$. (This does not require a lot of computation, just inspection of the form of the function $x(t)$.)

(d) Again assume that on Jan. 1, 2016 the population was 3000. What stocking rate will give a system response that is periodic (i.e. so that on any given day of the year the population is the same the next year on that same day.)

**Problem 5**  (Topic 1)  (14: 7,7)

Read the section G.2 (orthogonal trajectories) in the supplementary notes.

For each of the following families of curves,

  (i) find the ODE satisfied by the family
  (ii) find the orthogonal trajectories to the given family
  (iii) sketch the given family and the orthogonal trajectories.

(a) The family of curves $x^2 + 2y^2 = c^2$.

(b) The family of curves $y = ce^{-x}$.