ES.1803 Problem Set 2, Spring 2018
(due in class on Thursday, Sep. 20)

Part I  (30 points)

EP = Edwards & Penney; SN = Supplementary Notes (all have solutions)
Problems marked ‘Others’ are not to be handed in.

**Topic 4** (R, Sep. 13) Complex numbers and exponentials.
   Read: Class notes topic 4 or SN §C.
   Hand in: Part I problems: 4.1, 4.2b, 4.3, 4.4ab, 4.5 (posted with psets).

**Topic 5** (M, Sep. 17) Linear DEs, CC homogeneous case.
   Read: Class notes topic 5
   Hand in: Part I problems 5.1ab, 5.2a (posted with psets).

**Topic 6** (T, Sep. 18) Operators, inhomogeneous DEs, exponential response formula.
   Read: Class notes topic 6
   Hand in: SN Topic 6 part I problems: 1 (posted with the pset). Moved some problems
to next pset.

Continuation:  (W, Sep. 19) Discussion, review and catch up.

---

**Coming next**

**Topic 7** (R, Feb. 22) Inhomogeneous DEs: UC methods; Theory.
   Read: Class notes topic 7

---

**Part II  (100 points + 5 EC points)**

**Directions:** Try each problem alone for 20 minutes. If, after this, you collaborate, you
must write up your solutions independently.

**Problem 1  (Topic 4)  (15: 10,5)**
For this problem you might want to first spend 5 minutes playing with the Complex Roots
(a) For each of the following compute all three cube roots and plot them in the complex plane.
   (i) 1    (ii) $-i$

(b) Without computation (but with the applet if you like) describe the common pattern
   for $(1)^{1/5} \quad (-1)^{1/5}$ and $(i)^{1/5}$. (We’re looking for a short simple answer.)

**Problem 2  (Topic 4)  (10)**
The sinusoidal identity relates a sum of sinusoids in rectangular and polar (or amplitude
phase) form:

$$a \cos(\omega t) + b \sin(\omega t) = A \cos(\omega t - \phi),$$

with $a, b, A$ and $\phi$ given in the figure below.
Verify this identity.

**Problem 3** (Topic 5) (20: 5,5,5,5)
(a) Using the characteristic equation method show the general solution to \( x'' - 4x = 0 \) is

\[ x(t) = c_1e^{-2t} + c_2e^{2t}. \]

(b) Find \( c_1 \) and \( c_2 \) giving a solution to the DE in part (a) satisfying the indicated initial conditions: \( x(0) = 1, \ x'(0) = 2. \)
(c) Suppose it is known that \( \lim_{t \to \infty} x(t) = 0. \) What can be said about \( c_1 \) and \( c_2? \) Is it possible for \( x(t) \) to converge to any number other than zero as \( t \to \infty? \) Explain.
(d) Write down a second order homogeneous linear ODE which has \( e^{-2t} \) and \( -21 + e^{-2t} \) as solutions.

**Problem 4** (Topic 5) (30: 10,4,4,4,8)
Parts a-d of this problem deal with the equation \( x''' + x'' + x' + x = 0. \)
(a) Give the general real-valued solution to the equation.
(Hint: you should be able to guess one of the roots.)
(b) Describe all the real-valued periodic solutions.
(c) Describe all the solutions that go to 0 as \( t \to \infty. \)
(d) Describe the behavior of the general solution found in part (a) as \( t \) goes to \( \infty. \)
(e) Write down another third order CC linear DE with the following properties:
(i) Its characteristic polynomial, \( p(r) \) has integer coefficients.
(ii) \( p(r) \) has one real and two complex roots.
(iii) All solutions of the DE tend to 0 as \( t \to \infty. \)
Hint: start with the roots of the characteristic polynomial.

**Problem 5** (Topic 5) (25: 5,10,5,5)
(a) Given the second order equation \( mx'' + bx' + kx = 0, \) write down the condition (as a formula in \( m, b \) and \( k \)) for underdamping.
(b) For the underdamped case, show the solution has the form \( x = Ae^{-\frac{b}{2m}t} \cos(\omega_1 t - \phi). \)
(c) For the underdamped case, show that the times \( t \) where the graph of the solution has a peak satisfy \( \tan(\omega_1 t - \phi) = -\frac{b}{2m\omega_1}. \)
(d) Now, let \( T \) be the time between successive positive peaks (i.e. the pseudo-period), and let \( r \) be the ratio of the heights of two successive peaks (i.e., higher peak/lower peak).
Show that the damping constant \( b = 2m \ln(r)/T \).

Note: this gives us a way of finding the damping constant by measuring output.

(e) (optional, not for credit) Start the MIT Math Applet: [http://mathlets.org/mathlets/damped-vibrations/](http://mathlets.org/mathlets/damped-vibrations/) and try changing the parameter sliders until you get a feel for what each does, mathematically and in terms of the spring-mass system.

Now, set \( m = 1.0 \), \( b = 1.0 \), \( k = 4.0 \), \( x(0) = 0.75 \), and \( \dot{x}(0) = 1.0 \). Put the cursor in the main window and use the crosshairs to find the pseudo-period \( T \) and the height of two successive peaks. Now, use these values to estimate the damping constant \( b \) using the formula in part (b), and see how close you get to \( b = 1.0 \).

Problem 6  (Topic 6)  (0)

Problem 6 has been moved to the next pset.

Problem 7  (Topic 5)  (Extra credit 5)

In this problem we consider energy dissipation in a damped harmonic oscillator. We will show that the differential equation is consistent with the law of conservation of energy. To be concrete, you can think about a damped spring-mass system. Suppose we have mass \( m \), spring constant \( k \) and damping coefficient \( b \) then the DE modeling this system is

\[
m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0.
\]

Using your 8.01 knowledge, write the total energy of the system as kinetic + potential energy. Then use the DE to show that the rate energy is dissipated by the damping force equals the rate total energy is lost.