18.03 Problem Set 2, Fall 2017
(due in class on Thursday, Sep. 21)

Part I  (30 points)

EP = Edwards & Penney; SN = Supplementary Notes (all have solutions)
Problems marked 'Others' are not to be handed in.

**Topic 4** (W, Sep. 13) Complex numbers and exponentials.
Read: Class notes topic 4 or SN §C.
Hand in: Topic 4 part I problems: 1, 2b, 3, 4ab, 5 (posted with pset).

**Topic 5** (R, Sep. 14) Linear DEs, CC homogeneous case.
Read: Class notes topic 5
Hand in: SN 2C/1cd.

**Topic 6** (M, Sep. 18) Operators, inhomogeneous DEs, exponential response formula.
Read: Class notes topic 6
Hand in: SN 2F/1ef, 3d; Topic 6 part I problems: 1-4 (posted with the pset).

**Coming next**

**Topic 7** (W, Sep. 20) Inhomogeneous DEs: UC methods; Theory.
Read: Class notes topic 7

Read: SN §S (pp.0-1); Class notes topic 8.

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Part II  (83 points + 3 EC points)

**Directions:** Try each problem alone for 20 minutes. If, after this, you collaborate, you
must write up your solutions independently.

**Problem 1**  (Topic 4)  (10: 5,5)
For this problem you might want to first spend 5 minutes playing with the Complex Roots

(a) For each of the following compute all three cube roots and plot them in the complex plane.
(i) 1  (ii) $-i$

(b) Without computation (but with the applet if you like) describe the common pattern
for $(1)^{1/5}$, $(-1)^{1/5}$ and $(i)^{1/5}$. (We’re looking for a short simple answer.)

**Problem 2**  (Topic 4)  (6)
The sinusoidal identity relates a sum of sinusoids in rectangular and what we will call
amplitude phase form:

$$a \cos(\omega t) + b \sin(\omega t) = A \cos(\omega t - \phi),$$

with $a$, $b$, $A$ and $\phi$ given in the figure below.
Verify this identity.

Problem 3  (Topic 5)  (20: 4,6,5,5)
(a) Using the characteristic equation method show the general solution to \( x'' - 4x = 0 \) is
\[
x(t) = c_1 e^{-2t} + c_2 e^{2t}.
\]
(b) In each of the following cases, find \( c_1 \) and \( c_2 \) giving a solution to the DE in part (a)
satisfying the indicated initial conditions:
   (i) \( x(0) = 1, \ x'(0) = 2 \)   (ii) \( x(1) = 1, \ x'(1) = 0 \).
(c) Suppose it is known that \( \lim_{t \to \infty} x(t) = 0 \). What can be said about \( c_1 \) and \( c_2 \)? Is it
possible for \( x(t) \) to converge to any number other than zero as \( t \to \infty \)? Explain.
(d) Write down a second order homogeneous linear ODE which has \( e^{-2t} \) and \( -21 + e^{-2t} \)
as solutions.

Problem 4  (Topic 5)  (20: 6,3,3,3,5)
Parts a-d of this problem deal with the equation \( x''' + x'' + x' + x = 0 \).
(a) Give the general real solution to the equation.
(Hint: you should be able to guess one of the roots.)
(b) Describe all the real periodic solutions.
(c) Describe all the solutions that go to 0 as \( t \to \infty \).
(d) Describe the behavior of the general solution found in part (a) as \( t \) goes to \( \infty \).
(e) Write down another third order CC linear DE with the following properties:
   (i) Its characteristic polynomial, \( p(r) \) has integer coefficients.
   (ii) \( p(r) \) has one real and two complex roots.
   (iii) All solutions of the DE tend to 0 as \( t \to \infty \).
Hint: start with the roots of the characteristic polynomial.

Problem 5  (Topic 5)  (17: 5,5,3,4)
(a) Given the second order equation \( m x'' + b x' + k x = 0 \), write down the condition (as a
formula in \( m, b \) and \( k \)) for underdamping.
(b) For the underdamped case, show the solution has the form \( x = A e^{-\frac{b}{2m} t} \cos(\omega_1 t - \phi) \).
(c) For the underdamped case, show that the times \( t \) where the graph of the solution has
a peak satisfy \( \tan(\omega_1 t - \phi) = -\frac{b}{2m\omega_1} \).
(d) Now, let \(T\) be the time between successive positive peaks (i.e. the pseudo-period), and let \(r\) be the ratio of the heights of two successive peaks (i.e., higher peak/lower peak). Show that the damping constant \(b = 2m \ln(r)/T\).

Note: this gives us a way of finding the damping constant by measuring output.

(e) (optional, not for credit) Start the MIT Math Applet: [http://mathlets.org/mathlets/damped-vibrations/](http://mathlets.org/mathlets/damped-vibrations/) and try changing the parameter sliders until you get a feel for what each does, mathematically and in terms of the spring-mass system.

Now, set \(m = 1.0,\ b = 1.0,\ k = 4.0,\ x(0) = .75,\ \text{and}\ \dot{x}(0) = 1.0.\) Put the cursor in the main window and use the crosshairs to find the pseudo-period \(T\) and the height of two successive peaks. Now, use these values to estimate the damping constant \(b\) using the formula in part(b), and see how close you get to \(b = 1.0.\)

**Problem 6** (Topic 6) (10)

In this problem we will model a car running over an uneven surface. To simplify the picture we’ll assume the car has just one wheel as shown in the figure

![Diagram of a car and suspension system](image.png)

(a) As the wheel rolls over the road the height \(y(t)\) changes. This pushes on the suspension system which consists of a spring with spring constant \(k\) and a damper with damping constant \(b.\) Assuming the car has mass \(m\) (and the other components are massless) find a DE to model the vertical motion \(x(t)\) of the car.

(b) Suppose \(m = 1000\text{kg},\ k = 10^5\text{N/m},\ b = 0\) and \(y(t) = 0.1\cos(5t)\text{m},\) where \(t\) is in seconds. Find the steady periodic solution to your DE from part (a).

(In the undamped case, steady periodic solution is not well-defined. So, what we’ll take this to mean is ‘find the solution given by the sinusoidal response formula.’)

**Problem 7** (Topic 5) (Extra credit 3)

In this problem we consider energy dissipation in a damped harmonic oscillator. We will show that the differential equation is consistent with the law of conservation of energy. To be concrete, you can think about a damped spring-mass system. Suppose we have mass \(m,\) spring constant \(k\) and damping coefficient \(b\) then the DE modeling this system is

\[ m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0.\]

Using your 8.01 knowledge, write the total energy of the system as kinetic + potential
energy. Then use the DE to show that the rate energy is dissipated by the damping force equals the rate total energy is lost.