18.03 Problem Set 6 extra credit extra credit, Fall 2017
(due in class on Friday, Oct. 27)

Part II (Extra Credit Problems 24 points)
There are too many good problems to assign them all. Here are several you can do for extra credit.

Extra credit problem 1  (Topic 17)  (14: 3,3,3,5)
(a) This problem examines closed circulating two and three compartment systems. We’ll see that two compartment systems never oscillate, while three compartment systems can.

Consider the closed two-compartment system with flow rates and volumes shown at right. Let \( x_1 \) and \( x_2 \) be the amount of solute in tanks 1 and 2 respectively. By analyzing input and output to each tank, derive \( \text{but don’t solve} \) a system of DEs for this system.

(b) By analyzing input and output to each tank, derive the DEs for the amount of salt in each tank for the closed three-compartment system with the general volumes and flow rates shown.

(c) Use physical reasoning to answer the following questions. What is the long-range behavior in each system from part (a) and part (b)? Which eigenvalue is responsible for this? Which eigenvalue controls how fast the system goes (asymptotically) to its equilibrium state?

(d) (i) Show that the two-compartment system can never oscillate.
(ii) Show (by finding an example with numbers) that the three-compartment system can oscillate.

Extra credit problem 2  (Topic 17)  (10: 3,2,2,3)
In this problem we’ll look at a slightly different coupled spring-mass system.

We will use with the Javascript applet Coupled Oscillators:

http://mathlets.org/mathlets/coupled-oscillators/

You should start by playing with it. It’s pretty easy to figure out.

Now make sure the time \( t \) is set back to 0 and that the initial velocities are 0. (You can do this by switching on \( v_1 \) and \( v_2 \) and grabbing the end of the velocity indicator.) Set all the spring constants to \( k_1 = k_2 = k_3 = 1 \) and the masses to \( m_1 = 2, m_2 = 1.25 \).
(a) In this system a normal mode is one where both \(x_1\) and \(x_2\) are sinusoids with the same frequency. Find two normal modes on the Mathlet. Is one in sync and one 180° out of sync, as they are in the equal mass case? In each case, use the crosshairs and readout of coordinates on the Mathlet to measure the amplitudes of the two sinusoids. In each case, write \(A_1\) for the amplitude of the first mass and \(A_2\) for the amplitude of the second, and compute the ratio \(A_2/A_1\). Then measure the period of these sinusoids and record it.

(b) Now write down the equations of motion (\(m_1 x_1'' = \cdots, m_2 x_2'' = \cdots\)) and the 4 \(\times\) 4 “companion matrix.” Write the companion matrix in block form where the upper right is the 2 \(\times\) 2 identity.

(c) Find the eigenvalues using the trick in part I problem 17.1. Based on these eigenvalue, what periods do the normal modes to have? Compare with your measurements.

(d) Find the corresponding eigenvectors, write down the two normal modes. (I mean: write down the sinusoidal solutions for \(x_1\) and \(x_2\). ) Write them in the form \(\mathbf{x} = A \cos(\omega t - \phi) \mathbf{v}\), where \(\mathbf{v}\) is a constant vector and \(\omega\) is a positive number, and \(A\) and \(\phi\) can be anything. Determine the ratio \(A_2/A_1\) from this computation, and compare with your measurements.