ES.1803 Problem Set 6, Fall 2018
(due in class on Thursday, Oct. 25)

Part I (30 points)
EP = Edwards & Penney; SN = Supplementary Notes (all have solutions)
Problems marked ‘Others’ are not to be handed in.

**Topic 15** (R, Oct. 18) Linear algebra: transpose, inverse, determinant, inner products
- Read: Topic 15 notes. (Also see SN §LA.4.)
- Hand in: Part I problems 15.1 (posted with the pset).

**Topic 16** (M, Oct. 22) Linear algebra: eigenvalues, diagonal matrices, decoupling
- Read: Topic 16 notes
- Hand in: Part I problems 16.1ab, 16.2b (posted with the pset).

**Topic 17** (T, Oct. 23) Matrix methods of solving systems, the companion matrix.
- Read: Topic 17 notes or SN §§LS.1-3.
- Hand in: Part I problems 16.3a, 17.1ab, 17.2abc, 17.3bc, 17.4, 17.5 (posted with the pset).

Continuation: (W, Oct. 24) Discussion, review and catch up.

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**Coming next**

**T Oct. 30** Exam 2, covers topics 10-17 / psets 4-6

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Part II (102 points + 27 extra-credit)

Directions: Try each problem alone for 20 minutes. If, after this, you collaborate, you must write up your solutions independently.

**Problem 1** (Topic 15) (8: 5,3)
- (a) Compute
  \[
  \begin{bmatrix}
  0 & 0 & 0 & 1 \\
  0 & 0 & 1 & 0 \\
  0 & 1 & 0 & 0 \\
  1 & 0 & 0 & 0 \\
  \end{bmatrix}
  \]
  by swapping rows until you have the identity matrix.
- (b) The matrix in part (a) is called a *permutation* matrix. Explain why.

**Problem 2** (Topic 15) (10: 5,5)
- (a) Suppose \( A \) is a square matrix and the entries in every row of \( A \) add up to 0. Find a non-zero vector \( \mathbf{x} \) such that \( A\mathbf{x} = 0 \). Why does this show that \( \det A = 0 \)?
- (b) Now suppose that the entries of each row add up to 1. Why does this show that \( \det(A - I) = 0 \)? Does this mean that \( \det A = 1 \)? Explain why, or give a counterexample.

**Problem 3** (Topic 15) (20: 5,5,5,5)
In the statements below \( A \) and \( B \) are square matrices of the same size. If the statement is true, give a reason (using the facts about determinants listed in the class notes for topic 15). If it is false, provide a \( 2 \times 2 \) counterexample.
(i) If $AB$ is invertible, then both $A$ and $B$ are invertible.
(ii) $\det(A - B) = \det A - \det B$.
(iii) $\det(AB) = \det(BA)$.
(iv) $\det(aB) = a\det(B)$, where $a$ is a number and $aB$ means: $B$ with all entries multiplied by $a$.

**Problem 4** (Topic 16) (20: 10,10)

This problem will make use of the “Matrix Vector” Mathlet, from [http://mathlets.org/mathlets/matrix-vector/](http://mathlets.org/mathlets/matrix-vector/).

Open it up. You can control the blue input vector $v$ by dragging it around, or by means of the rectangular coordinate sliders along the edge of the graph (which also read out the $x$ and $y$ coordinates), or by adjusting its polar coordinates using sliders below. You can also set the entries of the matrix using sliders. The yellow vector is $Av$.

(a) You can guess at an answer to each of the following questions by studying the Mathlet. But you need to give reasons or computational explanations as well! We’ll use the notation $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for the matrix $A$.

(i) What sort of curve does $Av$ trace out as the $x$ coordinate of $v$ is changed?
(ii) What sort of curve does $Av$ trace out as the $a$ coefficient in the matrix is moved?
(iii) What sort of curve does $Av$ trace out as the $c$ coefficient in the matrix is moved?

(b) (i) Select the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$. Move the input vector $v$ around, and watch how it relates to the product $Av$. Record the coordinates of any eigenvectors you may encounter. (I could say “eigenlines,” because if any multiple of an eigenvector is again an eigenvector!) The corresponding eigenvalues is $\lambda$ such that $Av = \lambda v$. Estimate the eigenvalues corresponding to your eigenvectors.

(ii) Now compute the eigenvalues of this matrix, and find a non-zero eigenvector for each.

**Problem 5** (Topic 14) (10: 5,5) Suppose that $A$ is $100 \times 100$, and $A^2 = 0$.

(a) True or false: the column space of $A$ is contained in the null space of $A$. Explain why if true, or give a counterexample if false.

(b) What is the largest possible rank of $A$? Explain why the rank of $A$ is at most some number $r$, and give an example of a $100 \times 100$ matrix $A$ such that $A^2 = 0$ and having rank $r$.

**Problem 6** (Topic 17) (24 : 10,5,4,5)

Consider Example 13.7 in the class notes for topic 13 ([http://web.mit.edu/jorloff/www/18.03-esg/notes/topic13.pdf](http://web.mit.edu/jorloff/www/18.03-esg/notes/topic13.pdf)). This models the temperature in an insulated bar. Assume that the temperatures $E_L$ and $E_R$ are both 0, i.e. the homogeneous case.

(a) Call the coefficient matrix $A$. Find the eigenvalues of $A$. For each eigenvalue the corresponding eigenspace is one dimensional, find a non-zero eigenvector.

(b) Describe the corresponding normal modes. Specify initial conditions leading to each one.
(c) Is this system stable?

(d) Now divide the bar into 6 sections with temperatures $T_1, T_2, \ldots T_6$ respectively. Write a system of first order differential equations in matrix form modeling this system. ($E_L$ and $E_R$ are both still 0.)

**Problem 7** (Topic 17) (10 + 3 EC: 5,5,EC-3)

(a) Consider the unforced damped coupled spring system shown. The masses are $m_1$ and $m_2$; spring constants are $k_1$ and $k_2$; there is one damper with damping constant $c$; $x$ is the displacement of $m_1$ from its equilibrium position and $y$ is the displacement of $m_2$ from its equilibrium position. The damping force is proportional to the speed of the damper through its medium.

![spring-diagram]

Show that the system of DE's governing the behavior of the system is

$$m_1 \ddot{x} = -k_1 x + k_2 (y - x) + c(y' - x')$$
$$m_2 \ddot{y} = -k_2 (y - x) - c(y' - x').$$

(b) Find the companion system consisting of 4 first order equations. Write your answer in matrix form.

(c) (Extra credit) Now, let $m_1 = 2$, $m_2 = 1$, $k_1 = 4$, $k_2 = 2$, $c = 1$.

You probably don’t want to go on to find the eigenvalues and eigenvectors by hand. Well, we can make Matlab do it for us by using the $[V, D] = \text{eig}(A)$ command.

You can find basic instructions for Matlab and a short tutorial on eigenstuff on the class website. You can also learn about `eig` by typing `help eig` at the matlab prompt.

Use this to find the real solutions for $x$ and $y$. (Round to 2 decimal places in your answer.)

See extra credit problems posted alongside the pset.

*End of pset 6*