Gravity Waves: The Kelvin Wedge and Related Problems

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Gravity waves in water present many interesting phenomena that are not immediately intuitive. Certain applied mathematics and physics concepts that arise throughout the study of gravity waves, such as the method of stationary phase, can be used to analyze and explain a variety of situations. This paper discusses some of the basic mathematical concepts of wave analysis, and then applies these concepts to gravity waves in water. In particular, this paper works through the derivation of the angle that confines the wake behind a moving boat on deep water, known as the Kelvin wedge. Behavior of the waves near the border of this wake is also discussed. Finally, an overview is given of two related wedge problems.

1 An Introduction to Waves

1.1 A Monochromatic Wave

For a preliminary understanding of the behavior of waves, we investigate their behavior in a single dimension. In particular, we look at waves on a fluid interface, such as a surface of water. These waves are *dispersive*, which means that wave velocity depends on wavelength. There is thus a relationship between the velocity c, the wave vector k, and the angular frequency ω of the wave:

$$c = \tfrac{\omega}{k}$$

The angular frequency is defined as $\omega = 2\pi/\tau$, and the wave vector is defined as $k = 2\pi/\lambda$. The above expression is equivalent to writing

$$c = \frac{\lambda}{\tau}.$$

A one-dimensional, *monochromatic* (having a single wavelength and frequency) sinusoidal wave ζ, propagating in the x direction, can be expressed in complex variables as follows:

$$\zeta = |\zeta_k| e^{i\chi_k} e^{i(kx-\omega t)}$$

The amplitude of the wave is represented by $|\zeta_k| e^{i\chi_k}$, which can be complex, but the origin is generally chosen such that the phase shift χ_k vanishes, leaving us with a real amplitude.

1.2 Carrier Waves

More complicated waves can be represented as a sum of monochromatic waves. If the waves differ only slightly in wave vectors, they will form a beat pattern. If, for example, two waves with wave vectors k_1 and k_2 close to a reference k_o interact, the resulting wave can be represented as

$$\zeta = 2\zeta_o \cos(\Delta kx - \Delta\omega t) e^{i(k_o x - \omega_o t)}$$

for wave vectors and angular frequencies

$$k_{2,1} = k_o \pm \Delta k,\ \omega_{2,1} = \omega_o \pm \Delta\omega.$$

The resulting *carrier wave* moves with a *group velocity*

$$c_g = \left(\frac{\mathrm{d}\omega}{\mathrm{d}k}\right)_{k=k_o}.$$

In the case of gravity waves on deep water, $\omega \sim \sqrt{k}$ (discussed further in section 2.1), which implies

$$\frac{\mathrm{d}\omega}{\omega} = \frac{\mathrm{d}k}{2k}$$
$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{\omega}{2k} = \frac{1}{2}c.$$

Thus, in the case of gravity waves on deep water, the crests of the carrier wave move twice as fast as the crests of the component monochromatic waves.

1.3 Spacial Evolution of Carrier Waves

The sum of waves in a group can be represented as

$$\zeta = \textstyle\sum_k \zeta_k e^{i\chi_k} e^{i(k_o x - \omega_o t)}$$

for a phase angle χ_k that varies with x and t:

$$\chi_k\{x,t\} = \Delta k x - \Delta\omega t.$$

We can expand this around ω in powers of k:

$$\chi_k\{x,t\} = \{x - \left(\tfrac{d\omega}{dk}\right)_o t\}\Delta k - \tfrac{1}{2}\left(\tfrac{d^2\omega}{dk^2}\right)_o t(\Delta k)^2 - \ldots$$

With the expression for the group velocity in section 1.2, this becomes

$$\chi_k\{x,t\} = (x - c_{g,o})\Delta k - \tfrac{1}{2}\left(\tfrac{d^2\omega}{dk^2}\right)_o t(\Delta k)^2 - \ldots$$

The maximum amplitude of the group will occur when the waves are as close as possible to being completely in phase with each other. This occurs when the phase is stationary with respect to changes in k:

$$\tfrac{\partial\chi_k}{\partial(\Delta k)} = 0$$

If we use the method of dominant balance and ignore higher terms in the Taylor expansion, the derivative becomes

$$\tfrac{\partial\chi_k}{\partial(\Delta k)} = x - c_{g,o} = 0$$
$$x = c_{g,o}$$

Our dominant balance holds for a range of wavevectors that makes the second term small:

$$(\Delta k)^2 = \frac{1}{\left(\frac{dc_g}{dk}\right)_o t}$$
$$\Delta k = k_2 - k_1 = \left(\left(\tfrac{dc_g}{dk}\right)_o t\right)^{-1/2}$$

The velocities of the leading and trailing edges of the group are thus $c_{g,1}$ and $c_{g,2}$. If we define the width of the group as

$$L(x=0,t) \approx \frac{1}{k_2 - k_1}$$

then the rate of spreading of the group is

$$\frac{dL\{t\}}{dt} = |c_{g,2} - c_{g,1}| \approx (k_2 - k_1)|\frac{dc_g}{dk}|.$$

For gravity waves in deep water, the group velocity decreases with increasing wave vector:

$$\frac{dc_g}{dk} < 0$$

This implies that for large t, the wavelengths at the front of the wake are longest, as shown in Figure 1.

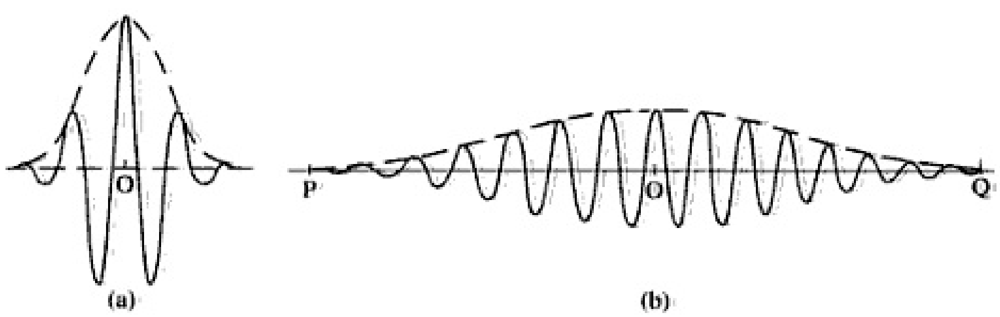

Figure 1: A dispersive wave group on a fluid interface that **(a)** starts as a Gaussian at time $t = 0$. **(b)** After traveling for some distance, there is a distribution of wavelengths over the length of the group. In this figure, $\frac{dc_g}{dk} > 0$, such that the longer wavelengths are near the back of the wake. This behavior is the opposite of that observed in deep water waves. [1]

2 Gravity Waves

2.1 Deep Water

The behavior of gravity waves depends on the boundary conditions applied to the problem. Boundary conditions can include restrictions on the wavelength, frequency, amplitude, and water depth, among other things. In this case, we are working up to the derivation of the Kelvin wedge, which is a phenomenon associated with ships in deep water. Thus, in this section we will primarily discuss gravity wave groups with a small amplitude compared to wavelength and confined to the surface of deep water. Mathematically, this can be expressed as

$$|\zeta| << \lambda << d$$

for a water depth d.

Working through the solution for these boundary conditions yields the following dispersion relation [1]:

$$\omega^2 = gk$$

The important part of this result, which is critical for the Kelvin wedge derivation, is the scaling relation $\omega \sim \sqrt{k}$. The velocity of each component of the wave group is thus

$$c = \tfrac{\omega}{k} = \sqrt{\tfrac{g}{k}} = \sqrt{\tfrac{g\lambda}{2\pi}}.$$

As discussed in section 1.2, this implies that the group velocity is half this:

$$c_g = \tfrac{1}{2}c = \tfrac{1}{2}\sqrt{\tfrac{g\lambda}{2\pi}}.$$

Working through the boundary condition mathematics gives the following form of for the vertical displacement of the wave [1]:

$$\zeta = \tfrac{ka}{i\omega}e^{i(kx-\omega t)}$$

for some real, non-time-dependent coefficient a. The physically significant part of this expression is the real part:

$$\mathrm{Re}\zeta = -\tfrac{ka}{\omega}\sin\left(kx - \omega t\right)$$

Thus, the wave is sinusoidal. Furthermore, according to our analysis in section 1.3, after the wave group has spread, the wavelength of the different parts of the group will be unequal. For gravity waves on deep water, the longer wavelengths will be at the leading edge of the wave group. When a storm creates a disturbance far out at sea, the first waves to reach the shore will have a longer wavelength than those that appear later on. [1]

2.2 Shallow Water

For the case of shallow water waves, the dispersion relation is instead

$$\omega^2 = gk\tanh(kd)$$

for a water depth d. When d is large, $\tanh(kd)$ asymptotes to 1, giving the dispersion relation for deep water in section 2.1. However, when d is very small, we use a linear approximation of the hyperbolic tangent: $\tanh(kd) \approx kd$. Our dispersion relation is thus

$$\omega^2 = gdk^2.$$

The important difference here is that the relation between the angular frequency and wave vector is now $\omega \sim k$ instead of $\omega \sim \sqrt{k}$. For this reason, the Kelvin wedge derivation below does not apply to waves on shallow water.

3 The Kelvin Wedge

The Kelvin wedge, first investigated by Lord Kelvin, is a phenomenon relating to the size of the wake that forms behind ships moving in deep water. We consider a ship moving to the left at a speed U, because by convention, wave groups travel in the positive x direction. Ships form a wave crest at the bow and a wave crest at the stern, leading to wave crests at a variety of angles α, as shown in Figure 2. Note that because the group velocity is half the phase velocity in this case, there can be no wave amplitude in front of the boat, despite appearances in the diagram. Since the bow sits at a wave crest, the speed of the wave crest at an angle α to the boat is found using trigonometry to be

$$c_\alpha = U \sin\alpha.$$

Using the dispersion relation for gravity waves in deep water, discussed in section 2.1, we can relate the wave vector to α:

$$c_\alpha = \sqrt{\tfrac{g}{k_\alpha}} = U\sin\alpha$$
$$k = \tfrac{g}{U^2\sin^2\alpha}$$

We now use the method of stationary phase to analyze the phase angle χ_α. The phase angle is given by

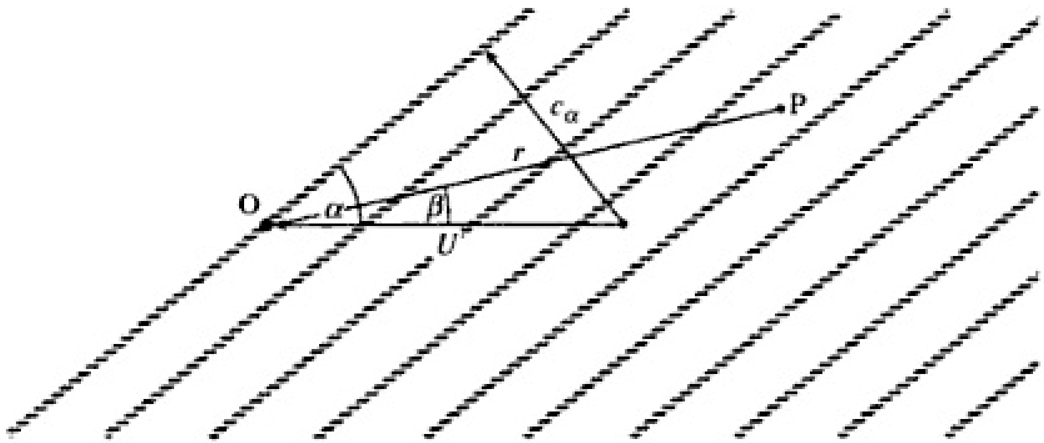

Figure 2: Schematic representation of a ship's bow, O, moving to the left at speed U. The ship creates wave crests at angles α from the bow. Wave interference is considered at an arbitrary point P at an angle β and distance r away from the bow. [1]

$$\chi_\alpha = \vec{k_\alpha} \cdot \vec{r}.$$

The phase angle changes rapidly over most of its domain, so the contributions of various waves cancel out in most places. The most significant contributions come from the area where the phase angle is changing negligibly with respect to the wave vector, when the derivative of the phase angle is equal to zero:

$$\tfrac{d\chi_\alpha}{dk_\alpha} = -r\sin(\alpha-\beta) - rk_\alpha \sin(\alpha-\beta)\tfrac{d\alpha}{dk_\alpha}$$

We can substitute in the derivative of k_α:

$$\tfrac{dk_\alpha}{d\alpha} = -2\cos\alpha \tfrac{g}{U^2\sin^3\alpha}$$
$$\tfrac{dk_\alpha}{k_\alpha} = -2\cos\alpha \tfrac{gd\alpha}{U^2\sin^3\alpha}\tfrac{U^2\sin^2\alpha}{g} = -2\tfrac{d\alpha}{\tan\alpha}$$
$$k_\alpha \tfrac{d\alpha}{dk_\alpha} = -\tfrac{1}{2}\tan\alpha$$

We plug this back into the expression for the derivative of the phase angle, and set the derivative equal to zero. This results in angles α_o around which there is constructive interference:

$$0 = -r\sin\alpha_o - \beta + \tfrac{1}{2}\tan\alpha_o r\cos\alpha_o - \beta$$
$$\sin\alpha_o - \beta = \tfrac{1}{2}\tan\alpha_o\cos\alpha_o - \beta$$
$$\tan\alpha_o - \beta = \tfrac{1}{2}\tan\alpha_o$$

We can simplify this expression further using the following two trigonometric identities:

$$\tan\Theta_1 + \Theta_2 = \tfrac{\tan\Theta_1+\tan\Theta_2}{1-\tan\Theta_1\tan\Theta_2}$$
$$\tan(-\beta) = -\tan\beta$$

Applying these identities, we can work through the following algebra:

$$\tfrac{\tan\alpha+\tan-\beta}{1-\tan\alpha\tan-\beta} = \tfrac{1}{2}\tan\alpha_o$$
$$\tfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} = \tfrac{1}{2}\tan\alpha_o$$
$$\tfrac{2(\tan\alpha_o-\tan\beta)}{\tan\alpha_o} = 1 + \tan\alpha_o\tan\beta$$
$$2 - \tfrac{2\tan\beta}{\tan\alpha_o} = 1 + \tan\alpha_o\tan\beta$$
$$1 = \tan\alpha_o\tan\beta + \tfrac{2\tan\beta}{\tan\alpha_o} = \tan\beta\left(\tan\alpha_o + \tfrac{2}{\tan\alpha_o}\right)$$
$$1 = \tan\beta\left(\tfrac{\tan^2\alpha_o+2}{\tan\alpha_o}\right)$$
$$\tan\beta = \tfrac{\tan\alpha_o}{2+\tan^2\alpha_o}$$

This result gives the angles β for which there is constructive interference (a wake behind the boat). There is no dependence on the distance r from the boat, the speed U of the boat, or any other parameters that relate to the boat, so this should hold for all boats at all uniform speeds in deep water! We can plot these angles over the full range of values of α_o, because the behavior is periodic. Figure 3 shows

$$\beta = \tan^{-1}\left(\tfrac{\tan\alpha_o}{2+\tan^2\alpha_o}\right)$$

plotted over the values of α_o between $-90°$ and $+90°$.

Figure 3 graphically demonstrates the result of the Kelvin wedge: there is a maximum value of β beyond which constructive interference does not occur in the wake. Doubling this maximum value of β gives the angular measure of the wake! We can calculate the size of the wake precisely. Note

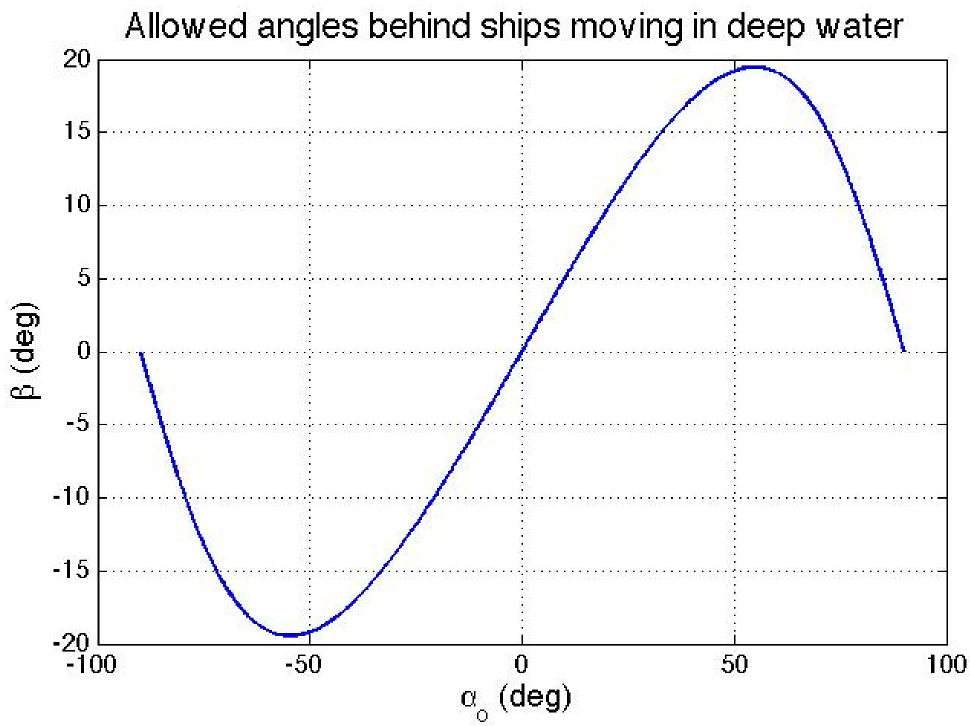

Figure 3: The range of angles β for which constructive interference occurs behind a boat moving in deep water. The allowed angles for constructive interference are from 0 to nearly 20° to either side of the boat's travel direction. Precise calculation shows this maximum deflection angle to be around 19.5°.

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that the maximum value of β and of $\tan\beta$ occur at the same value of α_o, so the derivative of $\tan\beta$ is taken to simplify the algebra.

$$0 = \frac{d(\tan\beta)}{d\alpha_o} = \frac{\sec^2\alpha_o}{2+\tan^2\alpha_o} - \frac{\tan\alpha_o}{(2+\tan^2\alpha_o)^2}(2\tan\alpha_o)\sec^2\alpha_o$$

$$\frac{1}{2+\tan^2\alpha_o} = \frac{2\tan^2\alpha_o}{(2+\tan^2\alpha_o)^2}$$

$$2+\tan^2\alpha_o = 2\tan^2\alpha_o \to 2 = \tan^2\alpha_o \to \tan\alpha_o = \sqrt{2}$$

$$\alpha_o = \pm\tan^{-1}\sqrt{2} \approx \pm 54.74°$$

The $\pm$ is introduced because we took the square root of the tangent. The size of half of the wake is the corresponding value of β. At $\alpha_o = \tan^{-1}\sqrt{2}$,

$\tan\beta = \frac{\sqrt{2}}{2+2} = \frac{\sqrt{2}}{2*2} = \frac{1}{2\sqrt{2}}$

$\beta = \tan^{-1}\left(\frac{1}{2\sqrt{2}}\right) \approx 19.47°$

Thus, the size of the Kelvin wedge is

$$2\tan^{-1}\left(\frac{1}{2\sqrt{2}}\right) \approx 38.94° \ !$$

This angular measure is the wake size behind an idealized boat moving in a straight line on deep water. The wedge shape can also be seen when you drag your finger through a bathtub at constant velocity, or when you dip your finger into a stream, allowing the water to flow past it [1]. An illustration of multiple wave crests forming a Kelvin wedge due to a source O can be seen in Figure 4.

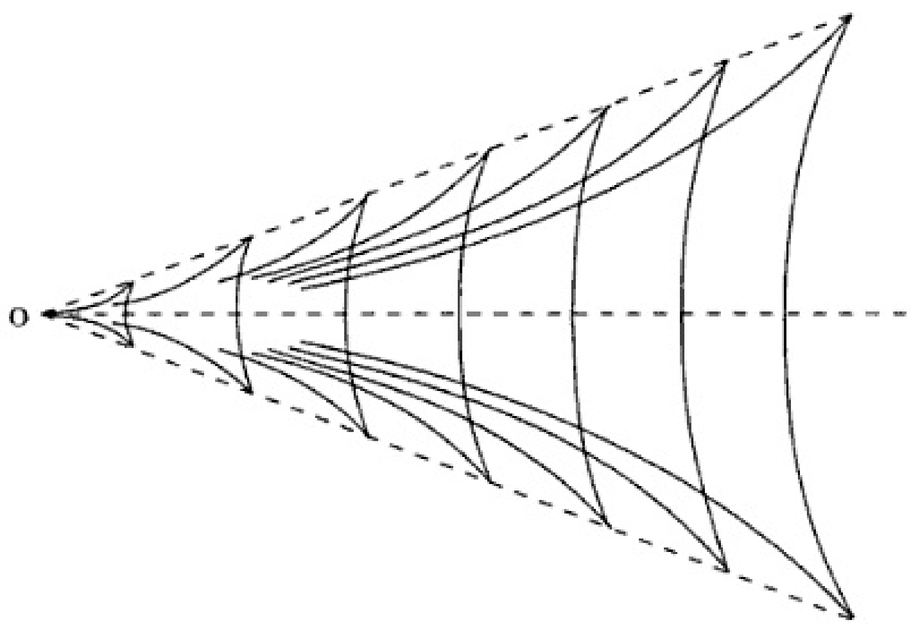

Figure 4: Wave crests in a Kelvin wedge. [1]

While this result is remarkable and non-intuitive, it is useful to consider the assumptions made in this calculation. It was assumed that the only sources of waves were from either a crest at the bow of the ship, or a trough at the stern of the ship. If waves are generated from both sources, as is usually the case, two overlapping Kelvin wedges will be seen. Furthermore, in a real world situation, some waves can be generated by the sides of a ship, causing further overlap of Kelvin wedge shapes. It should also be noted that we assumed that the amplitude of waves was small compared to the wavelength, as discussed in section 2.1. Should the amplitude become comparable to the wavelength, other non-linear phenomena will complicate the situation somewhat. [1] However, the Kelvin wedge can still be seen frequently in the real world.

4 Caustics and the Airy Function

While our above analysis works well for the area within the Kelvin wedge, it breaks down near the border of the wedge due to large local variations in wavenumber [2]. We thus have a transition region, known as a *caustic*, between an area with multiple wave groups and an area with no wave groups. Caustics occur at the edge of an area of trapped waves such as the Kelvin wedge. To describe the behavior of waves at this transition region, we must use the *Airy Function*.

We begin by defining the phase function of a typical wave group as

$$\psi(k) = \omega(k) - kx/t$$

for a wave group described by

$$\zeta = \int_{-\infty}^{\infty} F(k)\exp[it\psi(k)]dk = \int_{-\infty}^{\infty} F(k)\exp[i(kx - \omega(k)t]dk.$$

We can Taylor expand this integral around k_c, the wavenumber associated with the carrier waves at the caustic. The integral becomes

$$\zeta = \int_{-\infty}^{\infty} F(k_c)\exp\{it[\psi(k_c) + (k-k_c)\psi'(k_c) + \frac{(k-k_c)^2\psi''(k_c)}{2} + \frac{(k-k_c)^3\psi'''(k_c)}{6} + \ldots]\}dk.$$

At the caustic, the group velocity, $U = \omega'(k)$, has a minimum group velocity associated with a particular wavenumber k_c [2], at which point

$$U'(k_c) = \omega''(k_c) = 0 \Rightarrow \psi''(k_c) = 0.$$

We can thus remove the second derivative term, leaving us with the approximation

$$\zeta \approx \int_{-\infty}^{\infty} F(k_c)\exp\{it[\psi(k_c) + (k-k_c)\psi'(k_c) + \frac{(k-k_c)^3\psi'''(k_c)}{6}]\}dk.$$

with an error on the order of $|k-k_c|^4$ in the exponent.

To evaluate this integral, we must use the Airy function:

$$\mathrm{Ai}(X) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[i(sX + \frac{1}{3}s^3)]ds$$

The problem now lies in reducing our integral to a form in which we can use the Airy function. We do this using the following substitution:

$$\frac{s^3}{3} = t\frac{(k-k_c)^3\psi'''(k_c)}{6}$$
$$s = (k-k_c)(t\psi'''(k_c)/2)^{1/3}$$
$$(k-k_c) = s(t\psi'''(k_c)/2)^{-\frac{1}{3}}$$

This simplifies the cubic term to $is^3/3$. We can then simplify the linear term by defining X as follows:

$$sX = t(k-k_c)\psi'(k_c)$$
$$X = t(k-k_c)\psi'(k_c)\frac{(t\psi'''(k_c)/2)^{1/3}}{k-k_c} = t\psi'(k_c)[t\psi'''(k_c)/2]^{1/3}$$

The linear term is now isX. To change the variables of our integral, we take the derivative of s:

$$\frac{ds}{dk} = (t\psi'''(k_c)/2)^{1/3}$$
$$dk = (t\psi'''(k_c)/2)^{-1/3}ds$$

Thus, our integral has the form

$$\zeta = \int_{-\infty}^{\infty} F(k_c)\exp\{it\psi(k_c) + isX + is^3/3\}(t\psi'''(k_c)/2)^{-1/3}ds.$$

We can extract constant terms to simplify further:

$$\zeta = F(k_c)(t\psi'''(k_c)/2)^{-1/3}\exp[it\psi(k_c)]\int_{-\infty}^{\infty}\exp\{isX + is^3/3\}ds.$$

We have now reduced our integral to a form where we can use the Airy function, leaving us with

$$\zeta = F(k_c)(t\psi'''(k_c)/2)^{-1/3}\exp[it\psi(k_c)]\mathrm{Ai}(X)$$

for

$$X = t\psi'(k_c)[t\psi'''(k_c)/2]^{1/3}.$$

This analysis gives the general shape of how wave amplitude falls off at the edge of the Kelvin wedge (or any caustic boundary). The general shape of the Airy function, which in our case is amplified by the prefactors given in our form for ζ, is shown in Figure 5. Interestingly, the Airy function describes many phenomena in both water waves and other wave problems,

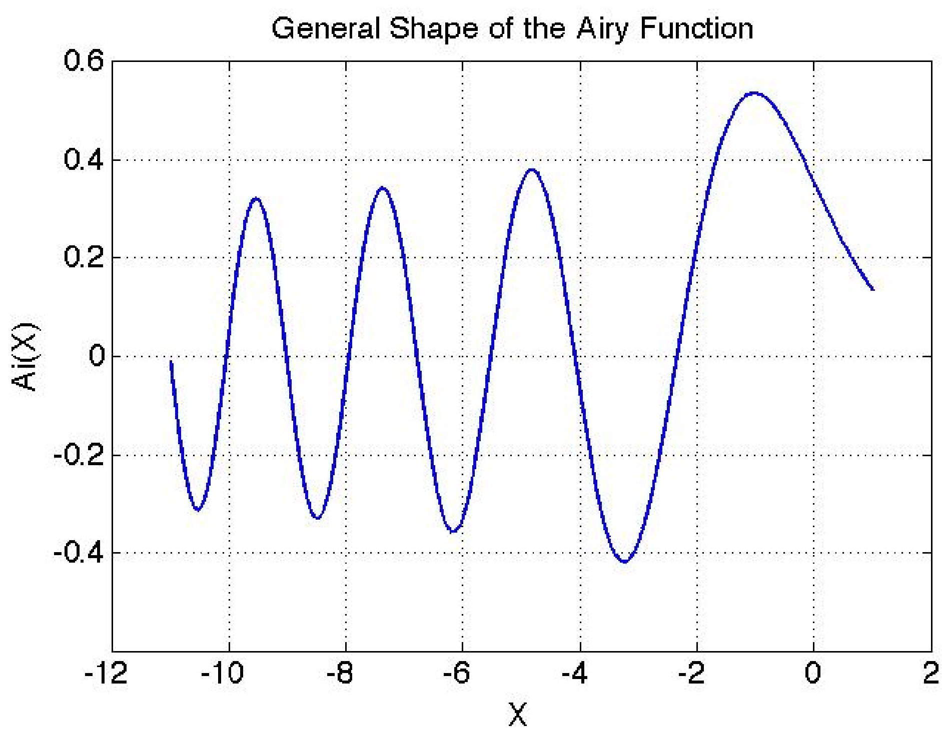

Figure 5: The general form of the Airy function, Ai(X). Beyond the Kelvin wedge and other caustics, wave amplitude falls off in a manner described by the Airy function.

such as optics. For example, the intensities of of primary, secondary, and tertiary rainbows are related to the relative peak amplitudes of the peaks of the Airy function [3].

5 Related Wedge Problems

There are many problems related to the waves created by "forcing effects" in both water and the air. The Kelvin wedge is the result of the specific case of a non-oscillating source traveling at a constant velocity in deep water, but all of these specifications can be altered to produce other situations. This section presents an overview of two problems investigated by M. J. Lighthill, but does not work through complete derivations of the results.

5.1 Gravity Waves from a Traveling Oscillating Source

One modification that can be made to the Kelvin wedge problem is to consider an oscillating source. Instead of a boat that steadily generates the crest of a wave at its bow, we consider a ship oscillating vertically with frequency σ_o. The Kelvin wedge is a special case of this problem where $\sigma_o = 0$.

For this problem, we will consider the gravity waves in reciprocal space, in which a plane wave can be described as

$$\phi = \phi_o \text{exp}\{i(-\sigma t + \vec{k} \cdot \vec{r})\} = \phi_o \text{exp}\{i(-\sigma t + lx + my + nz)\}.$$

The curves corresponding to different wave numbers are plotted in reciprocal space in Figure 6. The surfaces associated with each frequency can be expressed by

$$(\sigma_o + Ul)^4 = g^2(l^2 + m^2)$$

for speed U, gravitational acceleration g, and the x and y components of the wave vector in reciprocal space, l and m [4]. Since the plane wave is confined to the horizontal plane, there is no vertical component n in the equation. The form of the equation suggests two surfaces corresponding to each frequency, as seen in Figure 6. The wedge semi-angles that correspond to each surface are shown in Figure 7. This graph reveals that if the Kelvin wedge problem is modified to allow an oscillating source, a large range of wedge angles can be seen! The special case of the Kelvin wedge is at the intersection with the

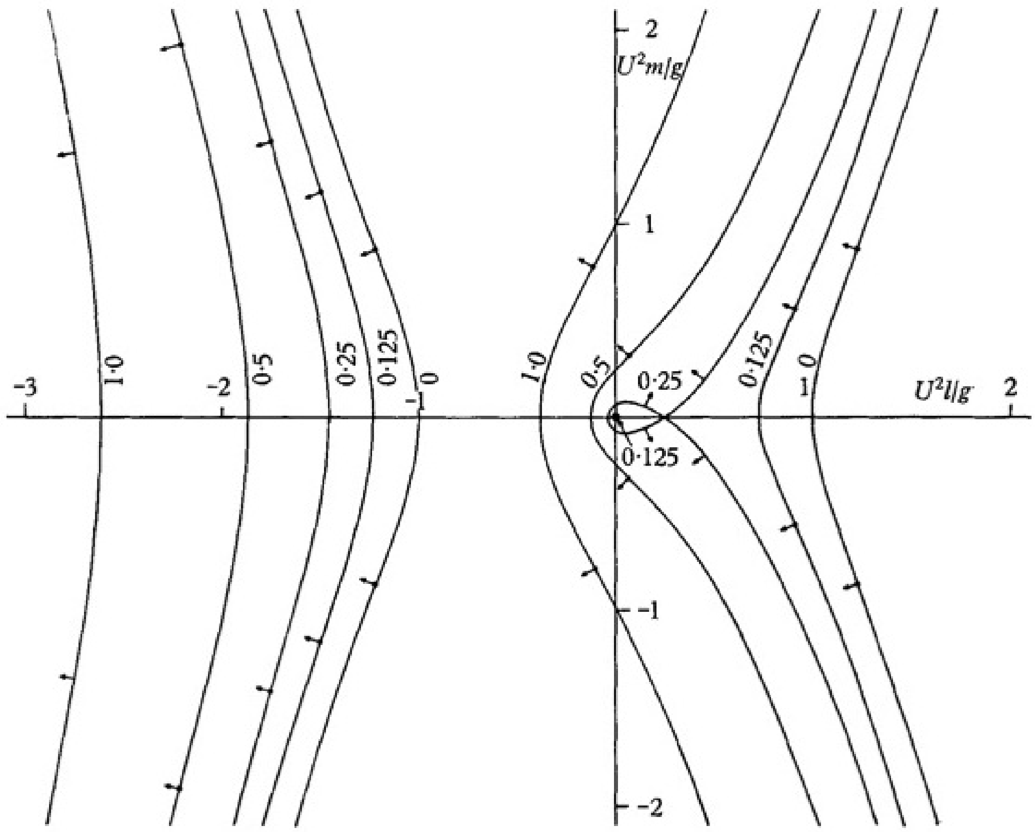

Figure 6: The curves associated with wave vectors from sources oscillating at various frequencies σ_o. Two curves are associated with each wave vector. The axes are U^2l/g and U^2m/g, the ratios between speeds, the x and y components of the wave vector in reciprocal space, and the gravitational acceleration. Frequencies are marked next to the curves. [4]

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vertical axis, where $U\sigma_o/g = 0$. At this point, the semi-angle is $\approx 19.47°$, as discussed in section 3.

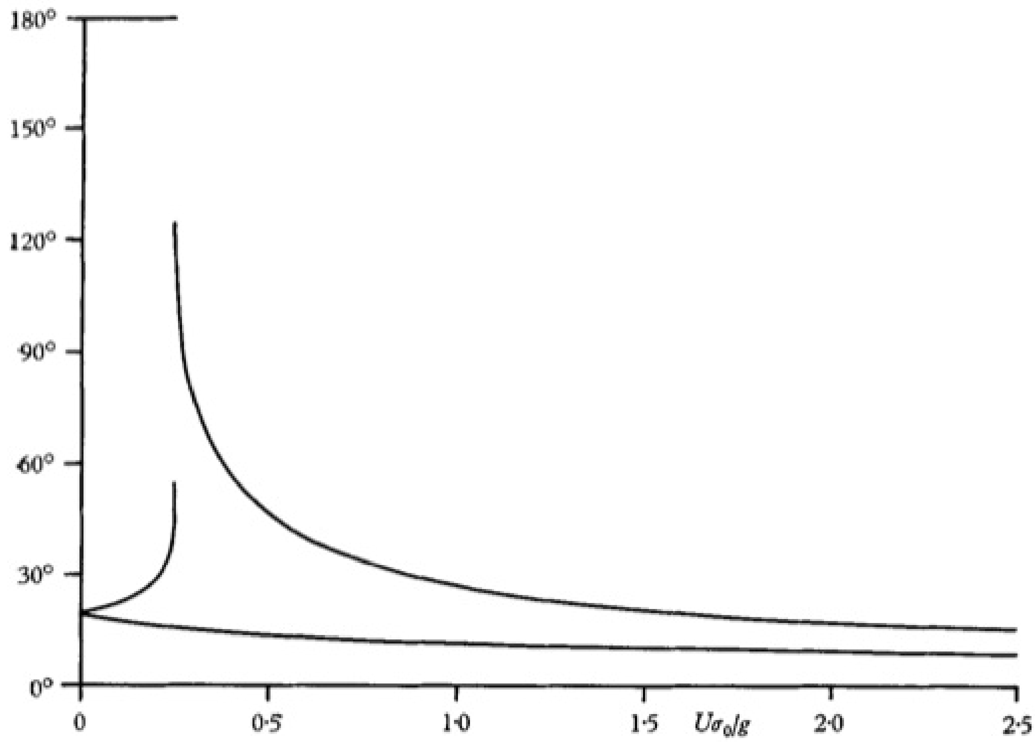

Figure 7: The semi-angles corresponding to the wedges produced by a source traveling and oscillating with frequency σ_o. The case of the Kelvin wedge corresponds to the point where $U\sigma_o/g$=0. [4]

5.2 Gravity Waves from a Vertically Moving Source

Another interesting wedge forms when the source, instead of moving along the surface of a body of water, moves vertically through the water. This case could describe the movement of a bubble or low-density material rising to the surface of water. The same analysis could apply to a *thermal*: a pocket of warm air rising through the atmosphere. When the uniformly-rising object moves, it creates internal gravity waves [4]. The surface of constant phase for this situation forms a wedge shape, as shown in Figure 8.

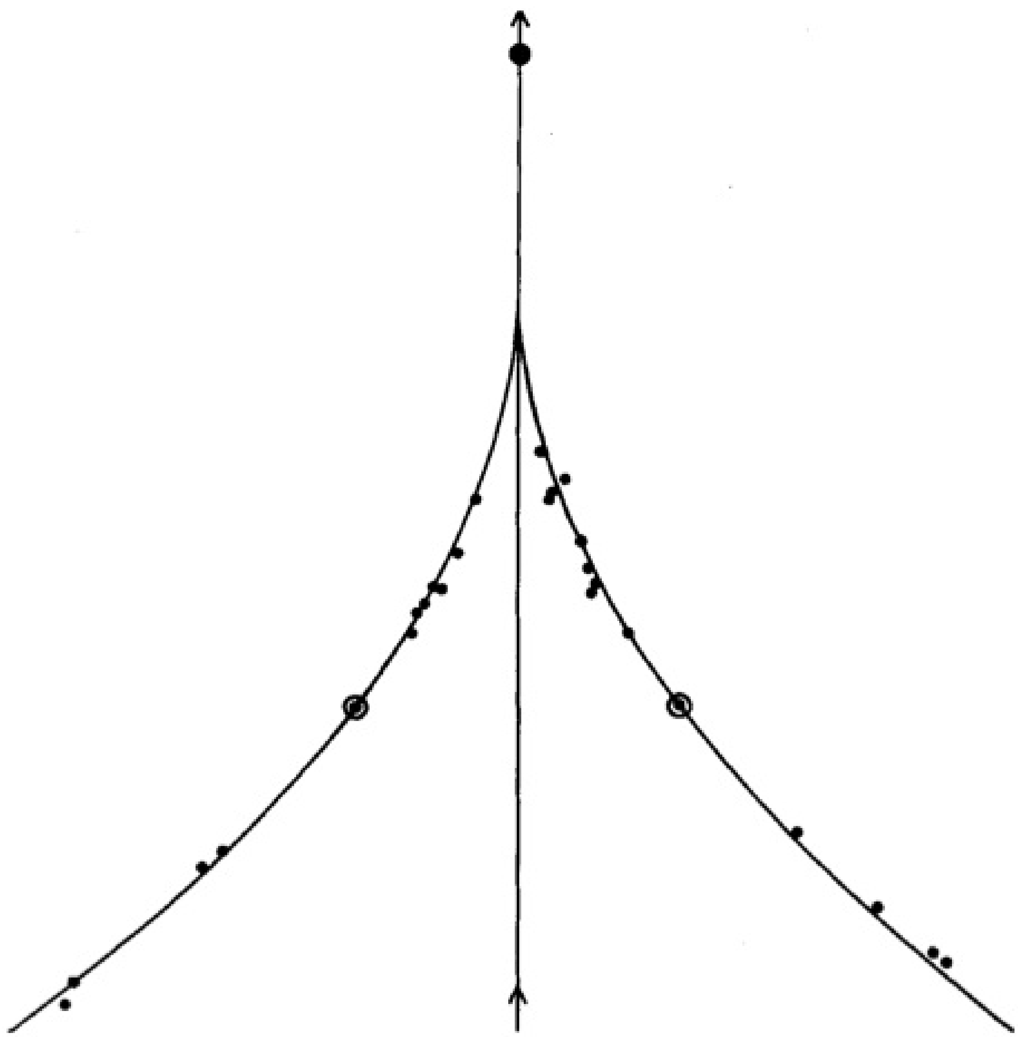

Figure 7: The surface of constant phase for the internal gravity waves created by an object moving uniformly upward through a uniform medium. The solid lines represent the shape of the theoretical curve, while the points are experimental data. [4]

6 Conclusion

The problems discussed in this paper are applications of the applied physics of waves as well as some of the mathematical methods used to analyze such problems. Some basic terminology was reviewed, and then the cases of gravity waves on deep and shallow water were considered. The method of stationary phase was used to derive the Kelvin wedge, a fascinating result describing the confinement of the waves produced a ship in deep water. The waves were found to be confined to a wedge of approximately $38.94°$. The border of this region is a caustic, at which the wave amplitude decays according to the Airy function. Finally two other wave problems relating to wedges were discussed.

Similar mathematical concepts can be used to explain the behavior of many complex situations, not only for gravity waves but for problems relating to eddies, drag flow, and high-atmosphere winds. Further interesting problems can be found in all of the references listed below. The problems discussed in this paper are just a few of the many problems in mathematics applied to fluids.

References

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