Estimating True Beliefs in Opinion Dynamics with Social Pressure

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***Abstract*—Social networks often exert social pressure, causing individuals to adapt their expressed opinions to conform to their peers. An agent in such systems can be modeled as having a (true and unchanging) *inherent belief* while broadcasting a *declared opinion* at each time step based on her inherent belief and the past declared opinions of her neighbors. An important question in this setting is *parameter estimation*: how to disentangle the effects of social pressure to estimate inherent beliefs from declared opinions. This is useful for forecasting when agents' declared opinions are influenced by social pressure while real-world behavior only depends on their inherent beliefs. To address this, Jadbabaie et al. [1] formulated the *Interacting Pólya Urn model* of opinion dynamics under social pressure and studied it on complete-graph social networks using an aggregate estimator, and found that their estimator converges to the inherent beliefs unless majority pressure pushes the network to consensus. In this work, we study this model on arbitrary networks, providing an estimator which converges to the inherent beliefs even in consensus situations. Finally, we bound the convergence rate of our estimator in both consensus and non-consensus scenarios; to get the bound for consensus scenarios (which converge slower than non-consensus) we additionally found how quickly the system converges to consensus.**

I. Introduction

Opinion dynamics studies how people's opinions evolve over time as they interact with others on social networks. This can provide insights and predictions about how public opinion develops on a variety of political, social, commercial and cultural topics, as well as guide marketing and political campaign strategies. For instance, Ancona et al. [2] used opinion dynamics models to study the spread of vaccine hesitancy and to develop marketing strategies to help combat it. Many common opinion dynamics models assume that people are truthful in the opinions they share. However, in reality this is not always the case, as people often alter their expressed views to better fit in with their social environment, which in turn feeds back into the social environment. This social pressure feedback loop can cause publicly-expressed opinions to become arbitrarily uniform over time [3], which poses difficulties in estimating and studying the underlying true public opinion.

In this work, we study an *Interacting Pólya Urn model* for opinion dynamics under social pressure, originating from [1] and developed further in [4], which captures a system of agents with stochastic behaviors who alter their publicly-expressed opinions to conform to their neighbors. This model consists of n agents on a fixed network communicating on an issue with two basic sides, denoted 0 and 1. Each agent i has an *inherent* (true and unchanging) belief ϕ_i, which is either 0 or 1, and a *bias parameter* γ_i indicating the ratio of the strength of their attachment to opinions 1 and 0, with $\gamma_i = 1$ indicating a neutral position (equal preference for both, though we will assume that no agents are neutral) and $\phi_i = 1 \iff \gamma_i > 1$ (higher preference for 1 than 0). The agents communicate their *declared opinions* to their neighbors at discrete time steps: at each integer step t, each agent i (simultaneously) declares an opinion $\psi_{i,t} \in \{0, 1\}$; the declarations of the agents at any t are random and independent, and each agent's probability of declaring 1 is determined by her bias parameter, inherent belief, and the opinions declared by her neighbors in the past. These terms are fully defined in Section II-B.

This can represent situations where agents alter their statements (contrary to their actual beliefs) to better fit in to their social environment – for instance, falsely signaling support for a political candidate they actually oppose. This is the primary motivation for the Interacting Pólya Urn model developed in [1]. In such a situation, forecasting the behavior of the agents (such as their votes) from social interactions requires separating their inherent beliefs from the pressure their social environment exerts on them. Furthermore, even though the agents' bias parameters offer a more complete and nuanced picture of their behavior, the true beliefs are often the key factor governing their behavior – for instance, voting or purchasing patterns.

The Interacting Pólya Urn model can also model situations where the agents honestly update their beliefs according to what they hear from others, but retain a bias towards their original beliefs.

A. Background Literature

Opinion dynamics has been a well-studied topic for many years with a number of mathematical models commonly used to capture specific social phenomena. Two especially important models are the DeGroot model [5], which seeks to model consensus formation, and the Friedkin-Johnsen model [6], which seeks to model social networks with persistent disagreements. Both models are fundamentally based on iterative averaging: each agent's opinion is represented as a real number, and at each step the agents all simultaneously update their opinion to a (weighted) average of their neighbors'

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opinions (including their own if the social network has self-loops); Friedkin-Johnsen models persistent disagreement by having each agent also include their own original opinion (which is fixed) in the averaging, thus avoiding consensus.

Many other models for opinions dynamics further build upon the Friedkin-Johnsen model, such as in [7], [8], [9]. Ye et al. [9] study a model in which each agent has both a private and expressed opinion, which evolve differently. Other models that look at opinion dynamics under social pressure include [10], [11] which consider dynamics similar to that in Hegselmann-Krause [12] and include parameters which measures an agent's resistance to change from their own belief. There are also stochastic models which focus on binary opinions including the voter model and some variations [13], [14]. The model in [3] features agents who pressure their neighbors into believing one of two opinions, and studies when the network cascades into total agreement. Other types of Pólya urn models have also been used to study contagion networks [15], [16].

The Interacting Pólya Urn model in [1] was developed to capture the case of agents who lie about their true beliefs in order to fit in with their social environment, i.e. at each step they each have an *inherent (true) belief* (known only to them) and a *declared opinion*, which may not agree. The mechanics of the dynamics was motivated by the well-known Bradley-Terry-Luce discrete choice model, in which an agent's declaration probability is proportional to the number of times she observes an opinion. The analysis in [1] is primarily focused on studying whether inherent beliefs are recoverable from unreliable declared opinions using an aggregate estimator, in order to estimate outcomes in cases where agents' declared opinions are influenced by social pressure while their actions depend only on their true beliefs (for instance, which candidate they vote for). This is carried out by establishing the convergence of the dynamics in the network and analyzing the equilibrium state, though the analysis is limited to the complete graph and all agents having the same amount of resistance to social pressure. In [4], the authors study the convergence properties of the Interacting Pólya Urn model introduced in [1] on arbitrary undirected networks, finding that the proportion of declared opinions of each agent converges almost surely to an equilibrium point in any network configuration. They also determined necessary and sufficient conditions for a network to approach consensus. We note that the definition of consensus used for the Interacting Pólya Urn model is that all agents declare a single opinion (all '0' or all '1') with probability tending to 1 as the process progresses.

B. Contributions

In [1], the authors consider when it is possible to asymptotically determine the inherent beliefs of the agents based on their history of declared opinions and those of their neighbors. They study a simplified case in which the social network is an (unweighted) complete graph and all agents have the same, known, degree of bias towards their inherent beliefs, and consider a specific aggregate estimator which tries to first estimate the proportion of agents with inherent belief 1 and then determine which agents those are. In this setting, they show that the aggregate estimator estimates the proportion of agents with inherent belief 1 if and only if the agents do not asymptotically approach consensus (where a large majority causes all agents declare the same opinion with probability approaching 1).

In this work, we consider the problem of estimating the agents' hidden parameters (inherent belief and resistance to social pressure) in the general setting presented in [4] where

1) the social network is an arbitrary weighted (and connected) undirected graph, possibly with self-loops;
2) the agents can have heterogeneous bias parameters, indicating different levels of resistance to social pressure or certainty in their inherent beliefs, which are not known. This addresses a limitation listed in [1].

(The focus of [4] was analyzing the asymptotic dynamics of the Interacting Pólya Urn model and thus [4] does not cover estimating inherent beliefs or bias parameters.)

Both the agents' inherent beliefs and bias parameters are unknown and must be inferred from observing the behavior of the network. This greatly increases the applicability of the model, as real-life social networks have a variety of different structures and people have varied reactions to social pressure.

In this setting, we study the maximum likelihood estimator (MLE), which estimates bias parameters from the history of declared opinions, rather than the aggregate estimator from [1]. We also derive a simplified estimator for inherent beliefs from the MLE, which takes a clean form with a low-dimensional sufficient statistic, consisting of two values which are simple to update at each step. We show that if the history of the agents' declared opinions is known, the MLE and the inherent belief estimator almost surely asymptotically converges to the correct inherent beliefs and bias parameters of all the agents in all such networks (even when the network approaches consensus). This resolves the fundamental question posed in [1] of whether such estimation is always possible.

We also show bounds on the convergence rate of the inherent beliefs estimator. These bounds are slower when the system approaches consensus, reflecting the loss of information in the declared opinions. We show that the convergence rate of the estimator in the case of consensus depends on the network structure through the largest eigenvalue of the normalized adjacency matrix multiplied by the bias parameters.

II. Model Description

We use the model from [4] which is a generalization of the model from [1]. Some changes from [1] include the addition that each edge in the network has a (nonnegative) weight denoting how much the two agents' declared opinions influence each other and the use of bias parameters instead of honesty parameters (which are different, but mathematically equivalent, representations of the same process).

A. Graph Notation

Let graph $G = (V, E)$ (undirected and including self-loops) be a network of n agents (corresponding to the vertices) labeled $i = 1, 2, \dots, n$. For each edge $(i, j) \in E$, there is a weight $a_{i,j} \geq 0$, where by convention $a_{i,j} = 0$ if $(i, j) \notin E$.

The matrix of these weights is $\boldsymbol{A} \in \mathbb{R}^{n\times n}$. The weighted degree of vertex i is $\mathsf{deg}(i) = \sum_j a_{i,j}$. The vector of degrees of all agents is

$$\boldsymbol{d} \triangleq [\mathsf{deg}(1), \mathsf{deg}(2), \ldots, \mathsf{deg}(n)]$$

and its diagonalization is $\boldsymbol{D} = \mathsf{diag}(\boldsymbol{d})$, i.e. the diagonal matrix of the degrees. Let the *normalized adjacency matrix* be $\boldsymbol{W} = \boldsymbol{D}^{-1}\boldsymbol{A}$ whose entries are $w_{i,j}$. We assume that G is connected so $\boldsymbol{W}$ is irreducible. We denote the largest eigenvalue of a matrix by $\lambda_{\max}(\cdot)$ (the matrices we use this with have real eigenvalues). Let $\mathbb{I}\{\cdot\}$ be the indicator function. Finally, we denote an all-0 vector as $\mathbf{0}$ and an all-1 vector as $\mathbf{1}$.

B. Inherent Beliefs and Declared Opinions

We define the Interacting Pólya Urn model of opinion dynamics under social pressure by defining the key parameters governing the behavior of the agents and their relationship to each other. The basic concept of the model is: each agent i declares at each (integer) step t an opinion $\psi_{i,t} \in \{0, 1\}$; in expectation, agent i imitates the (weighted) average opinion they have observed declared by their neighbors (including themselves via self-loops), but biased by an internal *bias parameter* $\gamma_i > 0$. The value of γ_i denotes how an observation of a neighbor declaring 1 is weighted compared to the same neighbor declaring 0, e.g. $\gamma_i = 2$ denotes that each observation of neighbor j declaring 1 counts twice as much as when they declare 0, while $\gamma_i = 1/2$ denotes the converse.[1]

Then, the inherent belief of agent i is the opinion they are biased toward:

$$\phi_i = \begin{cases} 1 & \text{if } \gamma_i > 1 \\ 0 & \text{if } \gamma_i < 1 \end{cases}$$

If $\gamma_i = 1$ then the agent is *unbiased* and is considered to not have an inherent belief; since the goal is to estimate the inherent beliefs of the agents, for the remainder of this work we assume that the agent under consideration is not unbiased. The inherent belief and bias parameter of agent i are assumed to be fixed and hidden from observers and other agents.

To formally state the model, let $b_i^0, b_i^1 > 0$ be the *initialization* of agent i's declared opinions, where $b_i^0 + b_i^1 = 1$. Then we define the *declared proportion* of 0's (or 1's) declared by agent i up to time $t \in \mathbb{Z}_+$ as:

$$\beta_i^0(t) = \frac{b_i^0}{t} + \frac{1}{t}\sum_{\tau=2}^{t}(1 - \psi_{i,\tau}) \tag{1}$$

$$\beta_i^1(t) = \frac{b_i^1}{t} + \frac{1}{t}\sum_{\tau=2}^{t}\psi_{i,\tau}\,. \tag{2}$$

Then $\beta_i^0(t), \beta_i^1(t) \in (0, 1)$ and $\beta_i^0(t) + \beta_i^1(t) = 1$; thus to specify these values it is sufficient to specify just $\beta_i(t) \triangleq \beta_i^1(t)$ (i.e. the time-averaged declarations of agent i's declared opinions with initial conditions).

[1]The honesty parameter in [1] is equivalent to the bias towards the agent's true belief, i.e. a honesty parameter of γ with a inherent belief of 0 corresponds to a bias parameter of $1/\gamma$.

For any agent i we also denote the total (weighted) proportion of opinions 0 and 1 she has observed by time t from her neighbors (including herself via self loop) as

$$\mu_i^0(t) = \frac{1}{\mathsf{deg}(i)}\sum_{j=1}^{n} a_{i,j}\beta_j^0(t) = \sum_{j=1}^{n} w_{i,j}\beta_j^0(t)$$

$$\text{and } \mu_i^1(t) = \frac{1}{\mathsf{deg}(i)}\sum_{j=1}^{n} a_{i,j}\beta_j^1(t) = \sum_{j=1}^{n} w_{i,j}\beta_j^1(t)\,;$$

as before, $\mu_i^0(t) + \mu_i^1(t) = 1$ by definition so it suffices to specify $\mu_i(t) \triangleq \mu_i^1(t)$. This corresponds to the social environment that agent i finds herself in at time t.

Then, at time $t+1$, each agent i will (independently) declare an opinion $\psi_{i,t+1}$ where

$$\psi_{i,t+1} \triangleq \begin{cases} 0 & \text{with probability } p_i(t) = f(\mu_i(t), \gamma_i) \\ 1 & \text{with probability } 1 - f(\mu_i(t), \gamma_i) \end{cases}.$$

and

$$f(\mu_i(t), \gamma_i) \triangleq \frac{\gamma_i\mu_i(t)}{\gamma_i\mu_i(t) + (1 - \mu_i(t))}\,.$$

The values of $\beta_i(t+1)$ and $\mu_i(t+1)$ for all i are updated according to the declared opinions at time t and the values of $\beta_i(t)$ and $\mu_i(t)$. Since $\mu_i(t) = \mu_i^1(t)$ and $1 - \mu_i(t) = \mu_i^0(t)$, this corresponds to weighting each observation of opinion 1 as γ_i times an equivalent observation of opinion 0.

We denote as $\mathcal{H}_t$ the *history* of the network up to time t (consisting of all declared opinions, including initializations, and thus can be used to compute $\beta_i(\tau), \mu_i(\tau)$ for $\tau \leq t$). The sequence $\mathcal{H}_0 \subseteq \mathcal{H}_1 \subseteq \ldots$ is also the notation we use for the filtration on which we can base the stochastic process. The random variables which generate the σ-algebras in this filtration are the declared opinions of all agents. We also denote the vectors of $\beta_i(t), \mu_i(t)$ over agents i as

$$\boldsymbol{\mu}(t) \triangleq [\mu_1(t), \ldots\mu_n(t)]^\top \text{ and } \boldsymbol{\beta}(t) \triangleq [\beta_1(t), \ldots\beta_n(t)]^\top\,.$$

In [4], it was shown that these dynamics must approach some equilibrium point satisfying

$$\beta_i = f(\mu_i, \gamma_i) \quad \text{for all } i \tag{3}$$

as $t \to \infty$ (with probability 1). In this work, we consider the following estimation problem (which was considered in [1] for a more restricted model on complete graphs): given the history $\mathcal{H}_t$ up to time t, can we estimate γ_i, ϕ_i for all agents i in the limit as $t \to \infty$?[2]

C. Consensus

An important term for this work is *consensus*, which needs to be defined appropriately for our stochastic system.

Definition 1. Consensus *is approached if*

$$\boldsymbol{\beta}(t) \to \mathbf{1} \text{ or } \boldsymbol{\beta}(t) \to \mathbf{0} \quad \text{as} \quad t \to \infty\,.$$

[2]While we assume for simplicity that b_i^0, b_i^1 are known to the estimator, this is not necessary for estimation in the long term as these terms become negligible in the limit as $t \to \infty$.

Since $\beta_i(t)$ represents the fraction (time-average) of agent i's declared opinions which are 1, consensus is approached when this ratio goes to 0 or 1. Let the diagonal matrix with $\boldsymbol{\gamma}$ along the diagonal be $\boldsymbol{\Gamma} = \mathsf{diag}(\boldsymbol{\gamma})$ and let $\boldsymbol{J_1} = \boldsymbol{\Gamma}^{-1}\boldsymbol{W}$ and $\boldsymbol{J_0} = \boldsymbol{\Gamma}\boldsymbol{W}$.

In [17], it is shown that consensus $\boldsymbol{\beta}(t) \to \mathbf{1}$ occurs when $\lambda_{\max}(\boldsymbol{J_1}) \leq 1$ and $\boldsymbol{\beta}(t) \to \mathbf{0}$ occurs when $\lambda_{\max}(\boldsymbol{J_0}) \leq 1$. Approaching consensus is important for the parameter estimation problem we consider in this work because it represents a major obstacle to solving the estimation problem, as it is an uninformative equilibrium.

D. Intuition for the Interacting Pólya Urn Model

In this section, we give another (intuitive) view of the Interacting Pólya Urn model. (For clearer intuition, we consider unweighted graphs, i.e. where $a_{i,j} = 0$ or 1.) Typically, urn models start with some composition of balls of different colors in an urn. At each step, a ball is drawn at random from the urn and additional balls are added based on the drawn ball according to some urn functions (thus affecting future draws). In the Interacting Pólya Urn model, at each step, agent i puts balls corresponding to her *neighbors'* declared opinions into her urn (the proportion of balls labeled 1 in agent i's urn is given by $\mu_i(t)$). Then, when agent i declares an opinion, it is modeled by the following: she draws a ball from her urn and declares the corresponding opinion; each ball corresponding with opinion 1 is γ_i times as likely to be drawn as one with opinion 0 (as given by function f). Note that if $\gamma_i = 1$ then agent i is simply (stochastically) mimicking the opinions her neighbors have declared in the past (plus her initial state, which becomes asymptotically negligible). We remark that the bias parameter is similar to the initial opinions in the Friedkin-Johnsen model [6] since they both are fixed parameters that influence all steps; however, note that there is a significant difference as the bias parameter can be overwhelmed over time by social pressure, thus leading to consensus.

E. Organization of results

In Section III we will introduce and define the inherent opinion and bias parameter estimators we will analyze; in Section IV we show that the estimators based on maximum likelihood are *consistent*, i.e. almost surely they converge to the correct result; and in Section V we will study the convergence rates of these estimators. In Section VI, in order to compute the convergence rate of the estimator in the case where the network approaches consensus, we determine a convergence rate result for the declare opinions.

Remark 1. *Our estimator consistency results from Section IV also apply to the Interacting Pólya Urn model on* directed *graphs, since the results are based only on the local environment of an individual agent. However, the convergence rate results in Sections V and VI depend on the convergence rate of the network to consensus, which has only been shown in the case of social networks on* undirected *graphs (see [4]); hence, those results do not generalize to directed graphs.*

III. Estimators for Inferring Inherent Beliefs and Bias Parameters

One of the key questions in [1] is whether it is possible to infer the inherent beliefs of agents from the history of declared opinions. The authors of [1] studied the Interacting Pólya Urn model on the complete graph using an aggregate estimator which keeps track of the fraction of declared opinions of all agents throughout time, and showed that this estimator may not converge to the inherent beliefs of all agents if they approach consensus. Consensus presents difficulties for estimators since asymptotically all agents approach the same behavior regardless of their inherent beliefs.

However, we show that estimators based on maximum likelihood estimation (MLE) almost surely infer the inherent belief of any agent i in the limit, even when consensus is approached. This fact is connected to [17, Lemma 2] – each agent declares both opinions infinitely often, yielding sufficient information to determine inherent beliefs over time.

Additionally, unlike [1], our formulation also allows agents to have different bias parameters. Thus, it is natural to ask how to estimate the bias parameter of any agent. Intuitively, after enough time has passed, the values of $\mu_i(t)$ and $\beta_i(t)$ will converge to values close to the equilibrium point. In such a case, we can use (3) to estimate the bias parameter γ_i and inherent belief ϕ_i with

$$\widehat{\gamma}_i^{eq}(t) = \frac{\beta_i(t)}{1-\beta_i(t)}\frac{1-\mu_i(t)}{\mu_i(t)} \tag{4}$$

$$\widehat{\phi}_i^{eq}(t) = \mathbb{I}\{\beta_i(t) < \mu_i(t)\} \tag{5}$$

These estimators are asymptotically consistent, i.e.

$$\lim_{t\to\infty} \widehat{\gamma}_i^{eq}(t) = \gamma_i \text{ and } \lim_{t\to\infty} \widehat{\phi}_i^{eq}(t) = \phi_i$$

when the dynamics converge to an interior equilibrium point (which is when consensus does not occur).

However, plugging the equilibrium values into (4) is not well-defined if $\beta_i(t)$ and $\mu_i(t)$ both converge to either 0 or 1 for all i, i.e. when consensus is approached. This shows that more careful analysis needs to be done in order to estimate the bias parameters and inherent beliefs in all circumstances. Simulations indicate that on the networks tested, the estimator in (5) tends to converge to the correct inherent belief, however, the number of time steps needed for the estimator to be correct with high probability can be very large in some situations.

A. Definition of Estimators

We assume at time t the estimator has at its disposal the history of agent i and agent i's neighbors' declarations up to and including time t (recall we denote this as $\mathcal{H}_t$). Given $\mathcal{H}_{t-1}$, we can compute exactly the values of

$$p_i(t) = \mathbb{P}\left[\psi_{i,t} = 1 | \mathcal{H}_{t-1}\right] = f(\mu_i(t-1), \gamma_i)\,.$$

Note that in general $\mathbb{P}\left[\psi_{i,t} = 1\right]$ is a random variable dependent on $\mathcal{H}_{t-1}$, while $\mathbb{P}\left[\psi_{i,t} = 1 | \mathcal{H}_{t-1}\right]$ is constant.

Our estimator to predict γ_i is based on the maximum log-likelihood estimator:

Definition 2. *The* single-step negative log-likelihood *for a given agent i at time $t > 1$ and parameter γ is*

$$\ell_i(\gamma, t) \triangleq -\Big(\mathbb{I}\{\psi_{i,t} = 1\} \log(f(\mu_i(t-1), \gamma)) + \mathbb{I}\{\psi_{i,t} = 0\} \log(1 - f(\mu_i(t-1), \gamma))\Big)$$

The negative log-likelihood for a given agent i at time t and parameter $\gamma \in (0, \infty)$ is

$$L_i(\gamma, t) \triangleq \sum_{\tau=2}^{t} \ell_i(\gamma, \tau)\,.$$

Here, γ_i is the actual bias parameter of agent i, whereas γ represents a proposed value whose loss we are measuring. The MLE for bias parameter γ_i gives the value of γ that maximizes the likelihood of agent i's declarations, which also minimizes the negative log-likelihood.

Definition 3 (Estimator for Bias Parameter)**.** *The* maximum likelihood estimator (MLE) *for the bias parameter γ at time t is given by*

$$\widehat{\gamma}_i(t) \triangleq \arg\min_{\gamma} L_i(\gamma, t)\,. \tag{6}$$

Since the inherent belief of an agent is defined as whether the bias parameter is greater than or less than 1, given the MLE estimator, we can always predict the inherent belief of agent i by taking $\mathrm{sign}(\log(\widehat{\gamma}_i(t)))$.

However, if we assume that $\gamma_i \neq 1$, and are only interested estimating the inherent beliefs, this reduces to a simpler form. Let $\bar{\beta}_i(t) = \frac{1}{t-1}\sum_{\tau=2}^{t} \mathbb{I}[\psi_{i,\tau} = 1]$, which a similar quantity to $\beta_i(t)$ except that the arbitrary initial conditions are not included. (If t is large, then difference between $\beta_i(t)$ and $\bar{\beta}_i(t)$ is negligible.)

Definition 4 (Inherent Belief Estimator)**.** *Let*

$$\widehat{\phi}_i(t) = \frac{1}{2}\mathrm{sign}\left((t-1)\bar{\beta}_i(t) - \left(\sum_{\tau=1}^{t-1} \mu_i(\tau)\right)\right) + \frac{1}{2}\,. \tag{7}$$

Multiplying by $1/2$ and adding $1/2$ maps the output of $\mathrm{sign}(\cdot)$ to 0 and 1. Fundamentally, this estimator requires only comparing

$$\bar{\beta}_i(t) \gtrless \frac{1}{t-1}\sum_{\tau=1}^{t-1} \mu_i(\tau)\,.$$

Note that $\widehat{\phi}_i(t)$ does not depend on knowing the bias parameter, as it only assumes that $\gamma \neq 1$, and the estimator is simple to compute as it only requires the aggregate count of an agent's declarations and her neighborhood's declarations.

Intuitively, this compares agent i's actual declarations against its expected declarations if $\gamma_i = 1$ (i.e. if the agent were unbiased); however, the consistency of this estimator is derived from that of the MLE for the bias parameter given in Definition 3. We show this derivation in Section IV-D.

In numerical simulations where the network does not approach consensus, the estimators $\widehat{\phi}_i(t)$ and $\widehat{\phi}_i^{eq}(t)$ perform similarly. Depending on the agent, either estimator may converge to the correct prediction faster. In networks where consensus is approached, the simulations suggest that both estimators will also eventually converge to the correct inherent belief with probability one, however, in certain networks, estimator $\widehat{\phi}_i^{eq}(t)$ may take significantly longer time to converge to all accurate predictions, as shown in Figure 2.

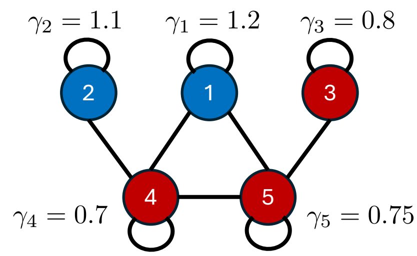

Fig. 1. Example network used in numerical simulations. Blue nodes have inherent belief $\phi_i = 1$ (i.e. $\gamma_i > 1$) and red have inherent belief $\phi_i = 0$ (i.e. $\gamma_i < 1$). All edges have weight 1. As per [4, Theorem 3], this network approaches a consensus of $\mathbf{0}$.

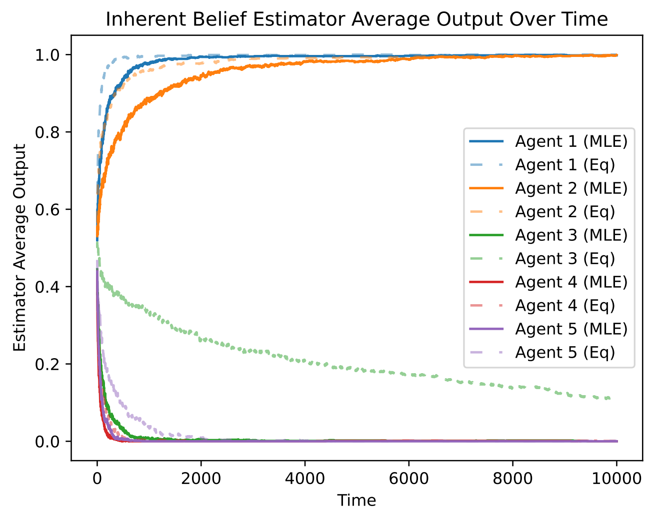

Fig. 2. Simulation for the network from Figure 1, comparing estimators for inherent beliefs. 'MLE' is the estimator (7) (solid) and 'Eq' is the estimator (5) (dashed). 1000 instances of the network were run; each line corresponds to the average prediction of the estimator at the given time over these instances. Note that for certain agents, the average of 'Eq' converges faster than 'MLE'; however, when given an agent in general agreement with its neighbors, such as Agent 3, 'Eq' can converge extremely slowly (dashed green line).

IV. Consistency of Estimators

In this section we show that the MLE based estimators for the bias parameter and inherent belief almost surely converge to the correct value. The key to showing this result is to first show that when two bias parameters, γ^* and $\gamma^\dagger$, are compared using log-likelihood ratios, in the long run, the true bias parameter (WLOG let this be γ^*) will have the higher likelihood. First, in Section IV-A we will examine preliminaries regarding log-likelihoods. Then, in Section IV-B we show that the difference of the log-likelihood ratios for γ^* and $\gamma^\dagger$ summed over time is a submartingale whose predictable quadratic variation is bounded by (a multiple of) its predictable expected value. This property allows us, in Section IV-C, to apply Freedman's inequality to get a concentration bound on the process which can be used to show that after some finite time γ^* will always have a higher likelihood than $\gamma^\dagger$. Then in Section IV-D, we show that how comparing just two bias parameters can be extended to show consistency of the bias estimator, and by extension the inherent belief estimator.

A. Preliminaries

1) Bounds on $\mu_i(t)$:

Lemma 1. *Letting $\kappa \triangleq \min_i(\min(b_i^0, b_i^1)) > 0$, for any agent i and time t,*

$$\mu_i(t) \in \left[\frac{\kappa}{t}, 1 - \frac{\kappa}{t}\right].$$

Proof. This follows since by definition $b_i^0, b_i^1 \geq \kappa$ for any i; thus by equations (1), (2) we know that $\beta_i^0(t), \beta_i^1(t) \geq \kappa/t$ so $\beta_i(t) = \beta_i^1(t) = 1 - \beta_i^0(t)$ satisfies $\beta_i(t) \in [\frac{\kappa}{t}, 1 - \frac{\kappa}{t}]$. But each $\mu_i(t)$ is a weighted average of $\beta_j(t)$, and hence $\mu_i(t) \in [\frac{\kappa}{t}, 1 - \frac{\kappa}{t}]$ for all i, t. □

Note that this means that any agent i will (almost surely) declare both 0 and 1 infinitely many times, even if the network approaches consensus, because either opinion has probability $\geq \Theta(1/t)$ at step t and $\sum_t 1/t = \infty$ (see [17]).

2) Negative Log-Likelihood Properties: We will analyze in depth the MLE which is key to our analysis. We start by introducing an alternative representation for $\ell_i(\gamma, t)$. Let $\tilde{\psi}_{i,t} = 2\psi_{i,t} - 1$, which takes values -1 and $+1$, instead of 0 and 1, giving a more symmetric representation of the process. Since $f(\mu_i(t), \gamma)$ is still the probability of $\tilde{\psi}_{i,t} = 1$,

$$\begin{aligned} \ell_i(\gamma, t) &= -\log\left(\frac{1}{1 + e^{-\tilde{\psi}_{i,t}\log\left(\gamma\frac{\mu_i(t-1)}{1-\mu_i(t-1)}\right)}}\right) \\ &= \log\left(1 + e^{-\tilde{\psi}_{i,t}\log\left(\gamma\frac{\mu_i(t-1)}{1-\mu_i(t-1)}\right)}\right). \end{aligned} \tag{8}$$

We reparameterize γ and $\mu_i(t)$ as follows:

$$\chi \triangleq \log\gamma \quad \text{and} \quad \nu_i(t) \triangleq \log\frac{\mu_i(t)}{1 - \mu_i(t)}.$$

Using χ symmetrizes the bias parameter across $\mathbb{R}$ (so $\chi = 0$ represents an unbiased agent). We thus define some quantities which take $\chi = \log\gamma$ as the argument instead of γ and use them where convenient:

$$\tilde{\ell}_i(\chi, t) \triangleq \ell_i(\gamma, t) \text{ and } \tilde{L}_i(\chi, t) \triangleq L_i(\gamma, t).$$

For this section to Section IV-C we will fix an agent i and then use γ^* and $\gamma^\dagger$ to represent any two possible choices for γ_i. We then show that if we know that one of these is the true value of γ_i, in the limit it is almost surely possible to determine which one (Theorem 2); this result will then be used to show that $\lim_{t\to\infty}\widehat{\gamma}_i = \gamma_i$ almost surely (Theorem 3). Define

$$Z(t) = Z(\gamma^*, \gamma^\dagger, t) \triangleq L_i(\gamma^\dagger, t) - L_i(\gamma^*, t). \tag{9}$$

If $Z(t)$ is positive, intuitively, γ^* fits the observed behavior better than $\gamma^\dagger$, so we expect γ^* to be the true parameter. Indeed, if γ^* is the true parameter, then

$$\begin{aligned} \mathbb{E}[Z(t)|\mathcal{H}_{t-1}] &= \sum_{\tau=2}^{t}\mathbb{E}\Bigg[\mathbb{I}\{\psi_{i,\tau} = 1\}\log\frac{f(\mu_i(\tau-1), \gamma^*)}{f(\mu_i(\tau-1), \gamma^\dagger)} \\ &\quad + \mathbb{I}\{\psi_{i,\tau} = 0\}\log\frac{1 - f(\mu_i(\tau-1), \gamma^*)}{1 - f(\mu_i(\tau-1), \gamma^\dagger)}\Big|\mathcal{H}_{\tau-1}\Bigg] \\ &= \sum_{\tau=2}^{t} D_{\text{KL}}(f(\mu_i(\tau-1), \gamma^*)\|f(\mu_i(\tau-1), \gamma^\dagger)) \end{aligned} \tag{10}$$

which is always a nonnegative quantity.

Proposition 1. *$L_i(\gamma, t)$ is a stochastic process which satisfies the following properties:*

(a) For fixed γ, $L_i(\gamma, t)$ (and $\tilde{L}_i(\chi, t)$) is an increasing function in t

(b) For fixed t, $\tilde{L}_i(\chi, t)$ is a strictly convex function in χ

(c) $\ell_i(\gamma, t) \in [0, \infty)$, and for a fixed t,
- *If $\tilde{\psi}_{i,t} = -1$, then $\ell_i(\gamma, t)$ is a decreasing function in γ (and $\tilde{\ell}_i(\chi, t)$ is decreasing in χ)*
- *If $\tilde{\psi}_{i,t} = 1$, then $\ell_i(\gamma, t)$ is an increasing function in γ (and $\tilde{\ell}_i(\chi, t)$ is increasing in χ)*

(d) If there exists $t_1, t_2 \leq t$ where $\tilde{\psi}_{i,t_1} = 1$ and $\tilde{\psi}_{i,t_2} = -1$, then $\tilde{L}_i(\chi, t)$ has unique finite minimum as a function in χ. Also $L_i(\gamma, t)$ has the same minimum at $\gamma = e^\chi$.

(e) For any $\gamma \neq \gamma_i$,

$$\mathbb{E}[\ell_i(\gamma, t)|\mathcal{H}_{t-1}] > \mathbb{E}[\ell_i(\gamma_i, t)|\mathcal{H}_{t-1}]$$

Proof.

(a) For each t, $\ell_i(\gamma, t)$ is nonnegative, so $L_i(\gamma, t)$ must be increasing. (Similar for $\tilde{L}_i(\chi, t)$).

(b)

$$\begin{aligned} \frac{d^2}{d\chi^2}\tilde{\ell}_i(\chi, t) &= \frac{d^2}{d\chi^2}\log\left(1 + e^{-\tilde{\psi}_{i,t}(\chi+\nu_i(t-1))}\right) \\ &= \frac{d}{d\chi}\frac{-\tilde{\psi}_{i,t}e^{-\tilde{\psi}_{i,t}(\chi+\nu_i(t-1))}}{1 + e^{-\tilde{\psi}_{i,t}(\chi+\nu_i(t-1))}} \\ &= -\tilde{\psi}_{i,t}\frac{d}{d\chi}\frac{1}{1 + e^{\tilde{\psi}_{i,t}(\chi+\nu_i(t-1))}} \\ &= \tilde{\psi}_{i,t}^2\frac{e^{\tilde{\psi}_{i,t}(\chi+\nu_i(t-1))}}{(1 + e^{\tilde{\psi}_{i,t}(\chi+\nu_i(t-1))})^2} \\ &= \frac{e^{\tilde{\psi}_{i,t}(\chi+\nu_i(t-1))}}{(1 + e^{\tilde{\psi}_{i,t}(\chi+\nu_i(t-1))})^2} \\ &> 0. \end{aligned} \tag{11}$$

Note that $\tilde{\psi}_{i,t}^2 = 1$. Thus $\tilde{\ell}_i(\chi, t)$ is convex for all t, and so $\tilde{L}_i(\chi, t) = \sum_{\tau=2}^{t}\tilde{\ell}_i(\chi, \tau)$ is also convex.

(c) Using (8), changing the sign of $\tilde{\psi}_{i,t}$ changes the sign on the exponent. If $\tilde{\psi}_{i,t}$ is positive, then the quantity in the exponent is decreasing as γ increases. The range is $[0, \infty)$ since the quantity in the log is greater than or equal to 1.

(d) Follows from (b) and (c). The function $\tilde{L}_i(\chi, t)$ must be convex and go to infinity at both ends. Since $L_i(\gamma, t) = \tilde{L}_i(\chi, t)$, it has the same minimum.

(e) Result follows from (10) setting $\gamma^* = \gamma_i$ and $\gamma^\dagger = \gamma$. The KL divergence must always be nonnegative and equal to zero iff $\gamma^* = \gamma^\dagger$.

□

□

B. Log-Likelihood Ratios and Martingales

To properly analyze the quantity (9), we need the following definitions. Unless otherwise stated, γ^* is the true parameter from which the random data is generated. The following definitions will be used starting from this section to Section IV-C. The *loss difference* is

$$Z(t) \triangleq Z(\gamma^*, \gamma^\dagger, t)$$
$$z(t) \triangleq z(\gamma^*, \gamma^\dagger, t) \triangleq \ell_i(\gamma^\dagger, t) - \ell_i(\gamma^*, t)\,.$$

The *predictable expected value* is

$$X(t) \triangleq X(\gamma^*, \gamma^\dagger, t) \triangleq \sum_{\tau=2}^{t} \mathbb{E}[z(\tau)|\mathcal{H}_{\tau-1}]$$
$$x(t) \triangleq x(\gamma^*, \gamma^\dagger, t) \triangleq \mathbb{E}[z(t)|\mathcal{H}_{t-1}]\,.$$

The *loss martingale* is

$$Y(t) \triangleq Y(\gamma^*, \gamma^\dagger, t) \triangleq X(t) - Z(t)$$
$$y(t) \triangleq y(\gamma^*, \gamma^\dagger, t) \triangleq x(t) - z(t)\,.$$

The *predictable quadratic variation* is

$$\begin{aligned} W(t) \triangleq W(\gamma^*, \gamma^\dagger, t) &\triangleq \sum_{\tau=2}^{t} \mathrm{Var}[z(\tau)|\mathcal{H}_{\tau-1}] \\ &= \sum_{\tau=2}^{t} \mathrm{Var}[y(\tau)|\mathcal{H}_{\tau-1}] \\ w(t) \triangleq w(\gamma^*, \gamma^\dagger, t) &\triangleq \mathrm{Var}[z(t)|\mathcal{H}_{t-1}] \\ &= \mathrm{Var}[y(t)|\mathcal{H}_{t-1}]\,. \end{aligned}$$

We give some preliminary results about these processes.

Proposition 2. *We have the following properties:*
(a) $Z(t)$ is a submartingale and $X(t)$ is strictly increasing
(b) $Y(t)$ is a martingale
(c) $W(t)$ is strictly increasing

Proof.

(a) The two statements are equivalent. From Proposition 1 (e), we have

$$x(t) = \mathbb{E}[\ell_i(\gamma^\dagger, t)|\mathcal{H}_{t-1}] - \mathbb{E}[\ell_i(\gamma^*, t)|\mathcal{H}_{t-1}] > 0\,.$$

(b) This follows from the definitions of $Y(t)$ and $y(t)$.

$$\begin{aligned} \mathbb{E}[y(t)|\mathcal{H}_{t-1}] &= \mathbb{E}[x(t) - z(t)|\mathcal{H}_{t-1}] \\ &= \mathbb{E}[\mathbb{E}[z(t)|\mathcal{H}_{t-1}] - z(t)|\mathcal{H}_{t-1}] = 0 \end{aligned}$$

and

$$\mathbb{E}[Y(t)|\mathcal{H}_{t-1}] = Y(t-1) + \mathbb{E}[y(t)|\mathcal{H}_{t-1}] = Y(t-1)\,.$$

(c) Because of the bounds on $\mu_i(t)$ given in Lemma 1, each $\ell_i(\gamma, t)$ must be non-constant so long as $\gamma \neq 1$. Then $z(t)$ is non-constant so long as $\gamma^* \neq \gamma^\dagger$. The quantity $w(t)$ is the conditional variance of $z(t)$ which therefore must always be positive. The quantity $W(t)$ is a sum of $w(t)$ so it must be increasing.

Next we determine bounds on our quantities. First we bound the predictable expected value $X(t)$. Since γ^* is the true bias, like in (10), we can write

$$x(t) = D_{\text{KL}}\left(f(\mu_i(t), \gamma^*) \| f(\mu_i(t), \gamma^\dagger)\right)\,. \tag{12}$$

Lemma 2. *For each time t, we can bound*

$$x(t) \geq \frac{(\sqrt{\gamma^*} - \sqrt{\gamma^\dagger})^2 \mu_i(t)(1 - \mu_i(t))}{\max\{\frac{\gamma^* + \gamma^\dagger}{2}, 1\}^2}\,.$$

Proof. To start, since $x(t)$ can be expressed as a KL divergence, we will lower bound this KL divergence by using squared Hellinger distance, specifically,

$$D_{\text{KL}}(P \| Q) \geq 2H^2(P, Q) \tag{13}$$

which we can derive from [18, 7.3]. For discrete distributions p and q over set $[1, \ldots, k]$,

$$H^2(p, q) = 1 - \sum_{i=1}^{k} \sqrt{p(i)q(i)}\,.$$

This gives that

$$\begin{aligned} &H^2(f(\mu, \gamma^*), f(\mu, \gamma^\dagger)) \\ &= 1 - \sqrt{\frac{\gamma^* \mu \gamma^\dagger \mu}{(\gamma^* \mu + (1-\mu))(\gamma^\dagger \mu + (1-\mu))}} \\ &\quad - \sqrt{\frac{(1-\mu)(1-\mu)}{(\gamma^* \mu + (1-\mu))(\gamma^\dagger \mu + (1-\mu))}} \\ &= 1 - \frac{\sqrt{\gamma^* \gamma^\dagger} \mu + (1-\mu)}{\sqrt{(\gamma^* \mu + (1-\mu))(\gamma^\dagger \mu + (1-\mu))}}\,. \end{aligned}$$

Let

$$\begin{aligned} A &= \sqrt{\gamma^* \gamma^\dagger} \mu + (1-\mu) \\ B &= \sqrt{(\gamma^* \mu + (1-\mu))(\gamma^\dagger \mu + (1-\mu))}\,. \end{aligned}$$

Note that $B > A$ since squared Hellinger distance is always between 0 and 1. Then

$$\begin{aligned} H^2(f(\mu, \gamma^*), f(\mu, \gamma^\dagger)) &= \frac{B - A}{B} \\ &= \frac{B^2 - A^2}{B(B+A)} \geq \frac{B^2 - A^2}{2B^2}\,. \end{aligned}$$

We can compute

$$\begin{aligned} &B^2 - A^2 \\ &= (\gamma^* \mu + (1-\mu))(\gamma^\dagger \mu + (1-\mu)) - (\sqrt{\gamma^* \gamma^\dagger} \mu + (1-\mu))^2 \\ &= (\gamma^* + \gamma^\dagger - 2\sqrt{\gamma^* \gamma^\dagger}) \mu (1-\mu) \\ &= (\sqrt{\gamma^*} - \sqrt{\gamma^\dagger})^2 \mu (1-\mu) \end{aligned}$$

and using AM-GM

$$\begin{aligned} B^2 &= (\gamma^* \mu + (1-\mu))(\gamma^\dagger \mu + (1-\mu)) \\ &\leq \left(\frac{\gamma^* + \gamma^\dagger}{2} \mu + (1-\mu)\right)^2 \leq \left(\max\left\{\frac{\gamma^* + \gamma^\dagger}{2}, 1\right\}\right)^2\,. \end{aligned}$$

This results in

$$H^2(f(\mu,\gamma^*), f(\mu,\gamma^*)) \geq \frac{(\sqrt{\gamma^*}-\sqrt{\gamma^\dagger})^2\mu(1-\mu)}{2\max\left\{\frac{\gamma^*+\gamma^\dagger}{2},1\right\}^2}.$$

Combining this with (13) and (12) completes the proof. ☐

Lemma 3. *If $\gamma^* \neq \gamma^\dagger$, then there is some $t_0 = t_0(\gamma^*,\gamma^\dagger)$ and $c_0 = c_0(\gamma^*,\gamma^\dagger) > 0$ such that for all $t > t_0$*

$$x(t) \geq c_0(\kappa/t).$$

Additionally, there are some constants k, t_1 (which depend on $t_0,\gamma^*,\gamma^\dagger$) such that for all $t > t_1$,

$$X(t) > kc_0\kappa\log(t).$$

The value of κ is defined in Lemma 1.

Proof. From Lemma 2, it suffices to let

$$c_0 = \frac{1}{2}\frac{(\sqrt{\gamma^*}-\sqrt{\gamma^\dagger})^2}{\max\{\frac{\gamma^*+\gamma^\dagger}{2},1\}^2}.$$

Then using Lemma 1,

$$\mu_i(t)(1-\mu_i(t)) \geq \frac{1}{2}\min\{\mu_i(t),1-\mu_i(t)\} \geq \frac{1}{2}\frac{\kappa}{t}.$$

This gives that $x(t) \geq c_0(\kappa/t)$ which implies

$$X(t) = \sum_{s=2}^{t} x(s) \geq \sum_{s=t_0}^{t} c_0\frac{\kappa}{s} \geq k_0c_0\kappa\log(t).$$

Constant k_0 accounts for some loss which occurs since $X(t)$ is a sum of terms $x(t)$, and for small t, the results may not be exact. ☐

The stochastic process $Z(\gamma^*,\gamma^\dagger,t)$ is a likelihood ratio test for determining whether γ^* or $\gamma^\dagger$ is the true parameter. Combining Lemma 3 and (10) gives that for $\gamma^* \neq \gamma^\dagger$,

$$\lim_{t\to\infty}\mathbb{E}[Z(\gamma^*,\gamma^\dagger,t)] = \lim_{t\to\infty}X(\gamma^*,\gamma^\dagger,t) = \infty. \quad (14)$$

The likelihood ratio $Z(\gamma^*,\gamma^\dagger,t)$ on average is very large as t gets large. This means that $Z(\gamma^*,\gamma^\dagger,t)$ can be used to distinguish which of the two parameters, γ^* or $\gamma^\dagger$, is the true parameter governing the data. If $Z(\gamma^*,\gamma^\dagger,t)$ is very large (positive), then γ^* is the true parameter. If $Z(\gamma^*,\gamma^\dagger,t)$ is very small (negative), then $\gamma^\dagger$ is the true parameter.

Remark 2. *The fact that $\mathbb{E}[Z(t)] \to \infty$ relies on $\mu_i(t) \in [\kappa/t, 1-\kappa/t]$, as discussed in Section IV-A1 (Lemma 1).*

If instead, $\mu_i(t)$ scales as $1/t^2$, then the limit of $\mathbb{E}[Z(t)]$ would be finite. In such a scenario, randomness might make $Z(t)$ unreliable for distinguishing between γ^* and $\gamma^\dagger$.

Changes to the model which may cause the condition $\mu_i(t) \in [\kappa/t, 1-\kappa/t]$ to fail include putting higher weight on previously declared opinions or having the network add more agents at each time step t.

Lemma 4. *For any t, we have*

$$|Z(t)-Z(t-1)| \leq \left|\log\frac{\gamma^*}{\gamma^\dagger}\right|$$

$$|Y(t)-Y(t-1)| \leq \left|\log\frac{\gamma^*}{\gamma^\dagger}\right|$$

Proof. Let $\mu = \mu_i(t)$. Since $\gamma^\dagger \neq \gamma^*$, we know that either:
(i) $\log\left(\frac{f(\mu,\gamma^*)}{f(\mu,\gamma^\dagger)}\right) < 0 < \log\left(\frac{1-f(\mu,\gamma^*)}{1-f(\mu,\gamma^\dagger)}\right)$ (if $\gamma^* < \gamma^\dagger$),
(ii) $\log\left(\frac{1-f(\mu,\gamma^*)}{1-f(\mu,\gamma^\dagger)}\right) < 0 < \log\left(\frac{f(\mu,\gamma^*)}{f(\mu,\gamma^\dagger)}\right)$ (if $\gamma^* > \gamma^\dagger$).
Since

$$Z(t) = \begin{cases} Z(t-1) + \log\left(\frac{f(\mu,\gamma^*)}{f(\mu,\gamma^\dagger)}\right) & \text{if } \psi_{i,t}=1 \\ Z(t-1) + \log\left(\frac{1-f(\mu,\gamma^*)}{1-f(\mu,\gamma^\dagger)}\right) & \text{if } \psi_{i,t}=0 \end{cases}$$

and by

$$\begin{aligned}&\left|\log\left(\frac{f(\mu,\gamma^*)}{f(\mu,\gamma^\dagger)}\right) - \log\left(\frac{1-f(\mu,\gamma^*)}{1-f(\mu,\gamma^\dagger)}\right)\right| \\ &= \left|\log\frac{\gamma^*}{\gamma^\dagger}\frac{\gamma^\dagger\mu+(1-\mu)}{\gamma^*\mu+(1-\mu)} - \log\frac{\gamma^\dagger\mu-(1-\mu)}{\gamma^*\mu+(1-\mu)}\right| \\ &= \left|\log\frac{\gamma^*}{\gamma^\dagger}\right|\end{aligned}$$

this means that $Z(t-1)$ and $Z(t)$ are both in the same $\left|\log\frac{\gamma^*}{\gamma^\dagger}\right|$-sized interval ($Z(t)$ is one of the endpoints and $Z(t-1)$ is somewhere in the middle). Additionally, $Y(t)$ (given history $\mathcal{H}_{t-1}$) is also a binary random variable whose possible outcomes are $\left|\log\frac{\gamma^*}{\gamma^\dagger}\right|$ apart, and since $\mathbb{E}[Y(t)\,|\,\mathcal{H}_{t-1}] = Y(t-1)$ this interval also must contain $Y(t-1)$, and we are done. ☐

Finally, we bound the predictable quadratic variation $W(t)$. Since $W(t)$ is defined as a variance, the following standard identity is helpful for finding an upper bound. Suppose that variable X takes two values, a with probability p and b with probability $(1-p)$, then

$$\mathrm{Var}[X] = p(1-p)(a-b)^2.$$

Applied to $w(t)$, we get that

$$\begin{aligned}&w(t) = \mathrm{Var}[z(t)|\mathcal{H}_{t-1}] \\ &= f(\mu_i(t),\gamma^*)\,(1-f(\mu_i(t),\gamma^*)) \\ &\quad\left(\log\frac{f(\mu_i(t),\gamma^*)}{f(\mu_i(t),\gamma^\dagger)} - \log\frac{1-f(\mu_i(t),\gamma^*)}{1-f(\mu_i(t),\gamma^\dagger)}\right)^2 \\ &= \left(\frac{\gamma^*\mu_i(t)}{1+(\gamma^*-1)\mu_i(t)}\right)\left(\frac{1-\mu_i(t)}{1+(\gamma^*-1)\mu_i(t)}\right) \\ &\quad\left(\log\frac{\gamma^*}{\gamma^\dagger}\frac{1+(\gamma^\dagger-1)\mu_i(t)}{1+(\gamma^*-1)\mu_i(t)} - \log\frac{1+(\gamma^\dagger-1)\mu_i(t)}{1+(\gamma^*-1)\mu_i(t)}\right)^2 \\ &= \frac{\gamma^*\mu_i(t)(1-\mu_i(t))}{(1+(\gamma^*-1)\mu_i(t))^2}\left(\log\frac{\gamma^*}{\gamma^\dagger}\right)^2. \quad (15)\end{aligned}$$

Lemma 5. *When $\gamma^* \neq \gamma^\dagger$, there exists a constant $c_1 = c_1(\gamma^*,\gamma^\dagger) > 0$ and t_1 such that for all $t > t_1$*

$$w(t) \leq c_1x(t).$$

This also implies

$$W(t) \leq c_1X(t). \quad (16)$$

Proof. Starting with (15), we have that

$$\begin{aligned}w(t) &\leq \frac{\gamma^*\left(\log\frac{\gamma^*}{\gamma^\dagger}\right)^2}{(\min\{1,\gamma^*\})^2}\mu_i(t)(1-\mu_i(t)) \\ &\leq \frac{\gamma^*\left(\log\frac{\gamma^*}{\gamma^\dagger}\right)^2}{(\min\{1,\gamma^*\})^2}\frac{\max\{\frac{\gamma^*+\gamma^\dagger}{2},1\}^2}{(\sqrt{\gamma^*}-\sqrt{\gamma^\dagger})^2}x(t)\end{aligned}$$

and thus we can set c_1 to be the coefficient in front of $x(t)$. Since $W(t)$ is a sum of $w(t)$ and $X(t)$ is a sum of $x(t)$, we naturally have $W(t) \leq c_1 X(t)$ for all t. $\square$

C. Concentration by Freedman's Inequality

We want to show that the test $Z(t) > 0$ works to distinguish whether γ^* or $\gamma^\dagger$ is the true parameter. We do this by showing that if γ^* is the true parameter, then almost surely $Z(t) \leq 0$ (i.e. the test fails) for only finitely many t. We show this by applying Freedman's inquality (this formulation taken from [19] (Thm 1.1), but originally from [20] (Thm 1.6)):

Theorem 1 (Freedman's Martingale Inequality [19])**.** *If $Y(t)$ is a martingale with steps $y(t) = Y(t) - Y(t-1)$ such that $|y(t)| \leq \alpha$ almost surely (and $Y(0) = 0$), and the* predictable quadratic variation *of $Y(t)$ is*

$$W(t) = \sum_{\tau=2}^{t} \mathbb{E}[y(\tau)^2 | Y(1), \ldots, Y(\tau-1)]$$

then for any $s, \sigma^2 > 0$,

$$\mathbb{P}[\exists\, t : Y(t) \geq s, W(t) \leq \sigma^2] \leq \exp\left(\frac{-s^2/2}{\sigma^2 + \alpha s/3}\right).$$

This inequality is an extension of Bernstein's inequality to martingales, where the variance of each step is not fixed but is itself a random variable dependent on the history. Using Theorem 1, we get our result for our test:

Theorem 2. *If $\gamma_i \in \{\gamma^*, \gamma^\dagger\}$, then the likelihood ratio test $Z(t) = L_i(\gamma^\dagger, t) - L_i(\gamma^*, t)$ is such that*

$$Z(t) \begin{cases} > 0 \text{ if } \gamma_i = \gamma^* \\ < 0 \text{ if } \gamma_i = \gamma^\dagger \end{cases}$$

for all but finitely many t.

Proof. For the proof, suppose that $\gamma_i = \gamma^*$. If $\gamma_i = \gamma^\dagger$, the same proof holds except with $-Z(t)$.

For the purposes of finding a contradiction, assume that $Z(t) \leq 0$ infinity often. Since $Z(t) = X(t) - Y(t)$, we have

$$Z(t) \leq 0 \iff X(t) \leq Y(t).$$

Using (16) from Lemma 5, we have

$$X(t) \leq Y(t) \iff W(t) \leq c_1 X(t) \leq c_1 Y(t).$$

Now suppose there are infinite values of t where $Z(t) \leq 0$. This means there are infinite values where $W(t) \leq c_1 Y(t)$. From (14), we know that $X(t) \to \infty$ as $t \to \infty$, implying that there is a t where $X(t) > 2$. In that case, we have

$$Y(t) - \frac{1}{2c_1} W(t) \geq X(t) - \frac{1}{2} X(t) \geq 1.$$

So long as t is large enough so that $X(t) > 2$, we have that

$$\begin{aligned} & Y(t) - \frac{1}{2c_1} W(t) \geq 1 \\ \implies & \exists s \in \mathbb{Z}_{>0} \text{ such that } \frac{1}{2c_1} W(t) \leq s \leq Y(t). \end{aligned} \tag{17}$$

Let us define a set of *bad times* where the estimator fails:

$$\mathcal{T} \triangleq \{t > 1 : Z(t) \leq 0, X(t) > 2\} = \{\tilde{t}_1, \tilde{t}_2, \ldots\}$$

where they are ordered $\tilde{t}_1 < \tilde{t}_2 < \ldots$. We assume that $\mathcal{T}$ is infinite and then derive a contradiction. We define

$$\tilde{s}_k \triangleq \max_s \left\{ s \in \mathbb{Z}_{>0} : \frac{1}{2c_1} W(\tilde{t}_k) \leq s \leq Y(\tilde{t}_k) \right\}.$$

From (17), we know such an $\tilde{s}_k$ exists for each k. In fact, $\tilde{s}_k = \lfloor Y(\tilde{t}_k) \rfloor$. If $\mathcal{T}$ is infinite, then this produces an infinite sequence of integers $\tilde{s}_1, \tilde{s}_2, \ldots$. Because $X(t) \to \infty$, which implies that $Y(\tilde{t}_k) \to \infty$, we have also that

$$\lim_{k \to \infty} \tilde{s}_k = \infty.$$

While the sequence $\tilde{s}_1, \tilde{s}_2, \ldots$ could have many copies of the same integer, the set $\{s : s = \tilde{s}_k \text{ for some } k\}$ must be infinite since $\lim_{k\to\infty} \tilde{s}_k = \infty$. In other words, if $\mathcal{T}$ is infinite there must be infinitely many positive integers s such that

$$Y(t) \geq s \text{ and } W(t) \leq 2c_1 s.$$

Applying Theorem 1, we get that there are infinitely many s such that

$$\begin{aligned} \mathbb{P}[\exists\, t : Y(t) \geq s, W(t) \leq 2c_1 s] &\leq \exp\left(\frac{-s^2/2}{2c_1 s + \alpha s/3}\right) \\ = \exp\left(\frac{s}{4c_1 + 2\alpha/3}\right) &= \exp(-\xi s) \end{aligned}$$

where $\xi = 1/(4c_1 + 2\alpha/3)$. Lemma 4 then gives the bound

$$y(t) \leq \left| \log \frac{\gamma^*}{\gamma^\dagger} \right| \triangleq \alpha.$$

For each $s \in \mathbb{Z}_{>0}$, let event A_s be

$$A_s = \{\exists t : Y(t) > s, W(t) \leq 2c_1 s\}.$$

Then,

$$\sum_{s=1}^{\infty} \mathbb{P}[A_s] \leq \sum_{s=1}^{\infty} \exp(-\xi s) < \infty$$

and therefore by Borel-Cantelli, almost surely only finitely many events A_s can occur. This is a contradiction and proves our result. $\square$

D. Combining Results to Prove Consistency of Estimators

The above concentration result shows that the MLE $\widehat{\gamma}_i(t)$ (6) converges asymptotically to the true γ_i.

Theorem 3. *For any agent i, almost surely,*

$$\lim_{t \to \infty} \widehat{\gamma}_i(t) = \gamma_i.$$

Proof. We take advantage of the alternative parameterization of $\chi = \log \gamma$ (from Section IV-A2). Let the MLE for χ be

$$\widehat{\chi}_i(t) = \arg\min_{\chi} \tilde{L}_i(\chi, t).$$

We will show that $\widehat{\chi}_i(t)$ converges to the true parameter, which we call χ_i, so $\widehat{\gamma}_i(t)$ converges to the true γ_i.

For any fixed $\epsilon > 0$, let $a = \chi_i - \epsilon$ and $b = \chi_i + \epsilon$. From Theorem 2, there exists some time t_a so that $\tilde{L}_i(a, t) >$

$\tilde{L}_i(\chi_i, t)$ for all $t > t_a$, and there exists some time t_b so that $\tilde{L}_i(b, t) > \tilde{L}_i(\chi_i, t)$ for all $t > t_b$.

At all times $t > \max\{t_a, t_b\} \triangleq t(\epsilon)$, the value of $\tilde{L}_i(\chi_i, t)$ is less than both $\tilde{L}_i(a, t)$ and $\tilde{L}_i(b, t)$. By Proposition 1(b), the function $\tilde{L}_i(\chi, t)$ is convex in χ, and thus the minimum of $\tilde{L}_i(\chi, t)$ at any $t > t(\epsilon)$ must be in $[a, b] = [\chi_i - \epsilon, \chi_i + \epsilon]$.

Thus, for every $\epsilon > 0$, we can always find a $t(\epsilon)$ where for all $t > t(\epsilon)$ we have that $\widehat{\chi}_i(t)$ is within ϵ of χ_i, and thus $\lim_{t\to\infty} \widehat{\chi}_i(t) = \chi_i$ completing the proof. □

This also shows that the inherent belief estimator from Definition 4 almost surely converges to the correct result.

Theorem 4. *Almost surely, if $\gamma_i \neq 1$, then*

$$\lim_{t\to\infty} \widehat{\phi}_i(t) = \phi_i$$

Proof. This is equivalent to

$$\lim_{t\to\infty} \left(\sum_{\tau=1}^{t-1} \mu_i(\tau)\right) - (t-1)\bar{\beta}_i(t) \begin{cases} < 0 & \text{if } \phi_i = 1 \\ > 0 & \text{if } \phi_i = 0 \end{cases}$$

The result follows from three facts:

(i) letting $\chi_i = \log(\gamma_i)$ and $\widehat{\chi}_i(t) = \arg\min_\chi \tilde{L}_i(\chi, t) = \log(\widehat{\gamma}_i(t))$ be the maximum likelihood estimator of χ_i, then $\lim_{t\to\infty} \widehat{\chi}_i(t) = \chi_i$;

(ii) for any t, $\tilde{L}_i(\chi, t)$ is strictly convex in χ;

(iii) $\left(\sum_{\tau=1}^{t-1} \mu_i(\tau)\right) - (t-1)\bar{\beta}_i(t) = \frac{\partial}{\partial\chi}\tilde{L}_i(\chi, t)\Big|_{\chi=0}$.

Fact (i) follows directly from Theorem 3 and (ii) is Proposition 1(b). Fact (ii) also shows that

$$\widehat{\chi}_i(t) > 0 \iff \frac{\partial}{\partial\chi}\tilde{L}_i(\chi, t)\Big|_{\chi=0} < 0\,;$$

and (assuming $\chi_i \neq 0$), $\phi_i = 1 \iff \chi_i > 0$. Thus facts (i) and (ii) show that (almost surely)

$$\begin{aligned} &\phi_i = 1 \implies \widehat{\chi}_i(t) > 0 \text{ for all sufficiently large } t \\ &\implies \frac{\partial}{\partial\chi}\tilde{L}_i(\chi, t)\Big|_{\chi=0} < 0 \text{ for all sufficiently large } t \end{aligned}$$

Thus, only fact (iii) remains to be shown.

Using $\tilde{L}_i(\chi, t) = \sum_t \tilde{\ell}_i(\chi, t)$ and

$$\tilde{\ell}_i(\chi, t) = \log\left(1 + e^{-\tilde{\psi}_{i,t}(\chi + \nu_i(t-1))}\right)$$

to evaluate the derivative (as in (11)) at $\chi = 0$, we get

$$\begin{aligned} \frac{\partial}{\partial\chi}\tilde{\ell}_i(\chi, t)\Big|_{\chi=0} &= \frac{-\tilde{\psi}_{i,t}}{e^{\tilde{\psi}_{i,t}\nu_i(t-1)} + 1} \\ &= \begin{cases} \frac{1}{\frac{1-\mu_i(t-1)}{\mu_i(t-1)}+1} & \text{if } \tilde{\psi}_{i,t} = -1 \\ \frac{-1}{\frac{\mu_i(t-1)}{1-\mu_i(t-1)}+1} & \text{if } \tilde{\psi}_{i,t} = 1 \end{cases} \\ &= \begin{cases} \mu_i(t-1) & \text{if } \tilde{\psi}_{i,t} = -1 \\ \mu_i(t-1) - 1 & \text{if } \tilde{\psi}_{i,t} = 1 \end{cases} \\ &= \mu_i(t-1) - \mathbb{I}\{\psi_{i,t} = 1\}\,. \end{aligned}$$

And thus the derivative of the entire negative log-likelihood evaluated at 0 is given by

$$\begin{aligned} \frac{\partial}{\partial\chi}\tilde{L}_i(\chi, t)\Big|_{\chi=0} &= \sum_{\tau=2}^{t} \mu_i(\tau-1) - \mathbb{I}\{\psi_{i,t} = 1\} \\ &= \left(\sum_{\tau=1}^{t-1} \mu_i(\tau)\right) - (t-1)\bar{\beta}_i(t)\,. \end{aligned}$$

This shows (iii) and completes the proof. □

V. Convergence Rates for Inferring Inherent Beliefs

While we have shown that the estimator for inherent beliefs in Definition 4 will eventually correctly converge to an agent's inherent belief, we are also interested in how fast it converges. Since the estimator $\widehat{\phi}_i(t)$ only takes values of 0 and 1 (and is therefore always exactly correct or exactly wrong) we say the estimator *converges by time t^** if

$$\widehat{\phi}_i(t) = \phi_i \text{ for all } t \geq t^*\,.$$

The question is: for any $\delta > 0$, how many steps t^* does it take for the estimator to have a $1 - \delta$ probability of converging? Or, in other words, for what t^* do we have

$$\mathbb{P}\left[\exists t \geq t^* : \widehat{\phi}_i(t) \neq \phi_i\right] \leq \delta\,?$$

In this section, the goal is to characterize how the estimator converges based on the convergence rate of declared opinions. Then in the remainder of this section, we determine bounds for the worst-case convergence rate. In the next section, we find bounds when declared opinions approach consensus. As in Section IV-C we fix the agent i and omit it from the notation in this section. Also as in Section IV-C, we assume that agent i has inherent belief $\phi_i = 1$ (so $\gamma_i > 1$). For agents where $\gamma_i < 1$, the same analysis holds replacing γ_i with $1/\gamma_i$.

The analysis for the convergence rate of the inherent belief estimator is similar to the analysis above for showing the convergence of the MLE. We use many of the same symbols in this proof as we did for the proof in the previous section (such as $X(t)$, $Y(t)$, etc.) but these will represent different (though analogous) quantities.

Note that $\widehat{\phi}_i(t) = \phi_i = 1$ if and only if

$$Z(t) \triangleq (t-1)\bar{\beta}_i(t) - \sum_{\tau=1}^{t-1} \mu_i(\tau) > 0\,.$$

Then $Z(t)$ is a stochastic process with differences

$$z(t) \triangleq Z(t) - Z(t-1) = \mathbb{I}\{\psi_{i,t} = 1\} - \mu_i(t-1)\,.$$

We then make a martingale $Y(t)$ as in Section IV-C: first, the expected updates and (cumulative) predictable expected value

$$x(t) \triangleq \mathbb{E}[z(t)|\mathcal{H}_{t-1}] \text{ and } X(t) \triangleq \sum_{\tau=2}^{t} x(\tau)\,.$$

We then derive $x(t)$ as

$$\begin{aligned} x(t) &= f(\mu_i(t-1), \gamma_i) - \mu_i(t-1) \\ &= \frac{(\gamma_i - 1)\mu_i(t-1)(1-\mu_i(t-1))}{1 + (\gamma_i - 1)\mu_i(t-1)} \end{aligned} \tag{18}$$

Note that since $\gamma_i > 1$ and $\mu_i(t-1) \in (0,1)$, we have $x(t) > 0$ for all t, and hence $X(t)$ is increasing and $Z(t)$ is a submartingale.

Proposition 3. *If $X(t) > g(t)$ for some increasing function g, and $X(t_0) > 2$ for some t_0, then*

$$\mathbb{P}\left[\exists t \geq t^* : \widehat{\phi}_i(t) \neq \phi_i\right] \leq \delta$$

holds for

$$t^* \geq \max\left\{g^{-1}\left(\frac{1}{\xi_i}\log\frac{1}{\delta(e^{\xi_i}-1)}\right), t_0\right\}$$
$$\xi_i = \frac{1}{4c_i + 2/3} \tag{19}$$

and $c_i = \frac{\gamma_i}{\gamma_i - 1}$.

Proof. First, note that g must be a monotonic function. We define

$$y(t) \triangleq x(t) - z(t) \text{ and } Y(t) \triangleq \sum_{\tau=2}^{t} y(\tau) = X(t) - Z(t)\,.$$

Then $Y(t)$ is a martingale, as $x(t) = \mathbb{E}[z(t)|\mathcal{H}_{t-1}] \implies \mathbb{E}[y(t)|\mathcal{H}_{t-1}] = 0$. Furthermore, the martingale $Y(t)$ has bounded step sizes:

$$\begin{aligned}|y(t)| &\leq \mathbb{E}[\mathbb{I}\{\psi_{i,t}=0\} - \mu_i(t-1)] \\ &\quad - (\mathbb{I}\{\psi_{i,t}=0\} - \mu_i(t-1)) \\ &= \mathbb{E}[\mathbb{I}\{\psi_{i,t}=0\}] - \mathbb{I}\{\psi_{i,t}=0\} \\ &\leq 1\end{aligned}$$

We define the predictable quadratic variation as:

$$w(t) \triangleq \mathrm{Var}[y(t)|\mathcal{H}_{t-1}] \text{ and } W(t) \triangleq \sum_{\tau=2}^{t} w(\tau)$$

Noting that (given $\mathcal{H}_{t-1}$) the value $\mu_i(t)$ is fixed yields

$$w(t) = \mathrm{Var}[\psi_{i,t}|\mathcal{H}_{t-1}] = \frac{\gamma_i\mu_i(t-1)(1-\mu_i(t-1))}{(1+(\gamma_i-1)\mu_i(t-1))^2}$$

This then implies bounds on the ratio between $w(t)$ and $x(t)$:

$$\begin{aligned}&\frac{w(t)}{x(t)} = \frac{\gamma_i}{\gamma_i-1}\frac{1}{1+(\gamma_i-1)\mu_i(t-1)} \in \left(\frac{1}{\gamma_i-1}, \frac{\gamma_i}{\gamma_i-1}\right) \\ &\implies \frac{1}{\gamma_i-1}x(t) \leq w(t) \leq \frac{\gamma_i}{\gamma_i-1}x(t)\end{aligned}$$

Since $c_i = \frac{\gamma_i}{\gamma_i-1}$, this gives

$$\begin{aligned}&w(t) \leq c_i x(t) \\ \implies &W(t) \leq c_i \sum_{\tau=2}^{t} x(\tau) \leq c_i X(t)\end{aligned}$$

Rewriting as $W(t)/c_i \leq X(t)$, we note that $X(t) \geq 2$ implies $W(t)/(2c_i) \leq X(t) - 1$; this means that when $X(t) \geq 2$, if $Y(t) > X(t)$ there must be some integer s such that $W(t)/(2c_i) < s < Y(t)$.

Let t_0 be when $X(t_0) \geq 2$. For any $t > t_0$, the estimator being wrong then implies

$$\begin{aligned}Z(t) < 0 &\iff X(t) - Y(t) < 0 \\ &\iff Y(t) > X(t) \\ &\implies \exists s \in \mathbb{Z}_+ : X(t) - 1 < s < Y(t) \\ &\implies \exists s \in \mathbb{Z}_+ : W(t)/(2c_i) < s < Y(t)\end{aligned}$$

Let event A_s for integer s be defined as

$$A_s = \{\exists t > t_0 : X(t) - 1 < s < Y(t)\}$$

If A_s occurs, we call s a *separator*; by the above, for any $t > t_0$, if the estimator is wrong at time t there must be a separator corresponding to t (note however that one separator s can work for multiple t). We now apply Freedman's Inequality (Theorem 1) to bound the probability that any given s is a separator:

$$\begin{aligned}\mathbb{P}[A_s] &= \mathbb{P}[\exists t > t_0 : X(t) - 1 < s < Y(t)] \\ &\leq \mathbb{P}[\exists t : Y(t) \geq s, W(t) \leq 2c_i s] \\ &\leq \exp\left(\frac{-s}{2(2c_i + 1/3)}\right)\,.\end{aligned}$$

However, given some s_0, we want to bound the probability that *any* integer $s > s_0$ is a separator. We thus define

$$B_{s_0} = \{\nexists\ s \in \mathbb{Z} > s_0 \text{ such that } A_s \text{ holds}\}\,.$$

Using (19) gives:

$$\begin{aligned}1 - \mathbb{P}[B_{s_0}] &= \sum_{s=s_0+1}^{\infty} \mathbb{P}[A_s] \leq \sum_{s=s_0+1}^{\infty} \exp(-\xi_i s) \\ &= \frac{e^{-\xi_i(s_0+1)}}{e^{\xi_i}-1}\,.\end{aligned}$$

If t is such that

$$s_0 + 1 \leq g(t) \leq X(t) \implies t \geq g^{-1}(s_0+1) \tag{20}$$

and B_{s_0} holds, then $Z(t) > 0$ and the estimator is correct. Thus, if (20) holds, then

$$\mathbb{P}[Z(t) < 0] \leq 1 - \mathbb{P}[B_{s_0}] \leq \frac{e^{-\xi_i(s_0+1)}}{e^{\xi_i}-1}\,.$$

If we want

$$\mathbb{P}[Z(t) < 0] \leq \frac{e^{-\xi_i(s_0+1)}}{e^{\xi_i}-1} \leq \delta$$

then

$$\begin{aligned}&e^{-\xi_i(s_0+1)} \leq \delta(e^{\xi_i}-1) \\ \implies &s_0 + 1 \geq \frac{1}{\xi_i}\log\frac{1}{\delta(e^{\xi_i}-1)}\,.\end{aligned}$$

Combining this with (20) gives

$$g(t) \geq \frac{1}{\xi_i}\log\frac{1}{\delta(e^{\xi_i}-1)}\,.$$

□

To use Proposition 3, we need to determine how quickly $\mu_i(t)$ or $\beta_i(t)$ approaches 0. Before doing a calculation of this, we show what happens if we use a worst-case bound on $\mu_i(t)$. Since $\mu_i(t-1) \in [\kappa/t, 1-\kappa/t]$, we have

$$x(t) \geq \frac{\gamma_i - 1}{\gamma_i} \frac{1}{2} \frac{\kappa}{t}$$

as $\mu_i(t)(1-\mu_i(t)) \geq \frac{1}{2}\min(\mu_i(t), 1-\mu_i(t))$. This implies that for $t \geq 3$,

$$X(t) \geq \frac{\gamma_i - 1}{\gamma_i} \frac{1}{4} \kappa \log(t)\,. \tag{21}$$

Using (21), we know that $X(t) \geq 2$ when

$$t \geq e^{8\frac{\gamma_i}{\gamma_i - 1}\kappa^{-1}} \triangleq t_0\,.$$

Then the estimator has converged with probability $\geq 1-\delta$ for all t such that

$$\frac{\gamma_i - 1}{4\gamma_i}\kappa \log(t) \geq \frac{1}{\xi_i} \log \frac{1}{\delta(e^{\xi_i} - 1)}$$
$$\implies t \geq \left(\frac{1}{\delta(e^{\xi_i} - 1)}\right)^{\frac{4}{\kappa}\frac{\gamma_i}{\gamma_i - 1}\left(4\frac{\gamma_i}{\gamma_i - 1} + \frac{2}{3}\right)} = \Theta\left((1/\delta)^c\right) \tag{22}$$

where c is a constant depending on κ and γ_i. Thus, we know that for some t^* on the order of $(1/\delta)^c$, the inherent belief estimator converges by time t^* with probability at least $1-\delta$.

However, since (21) is a lower bound, corresponding to using the lower bound of $\Theta(1/t)$ for $\mu_i(t)$, the computed convergence rate (22) is too low. This raises the question of improving it by using better bounds on $\mu_i(t)$, thus yielding a better bound of $X(t)$ in (21). This can be divided into two cases: consensus and non-consensus.

If the system does not approach consensus, then $X(t)$ is linear since for large enough t, $x(t)$ will be very close to a constant, and thus $X(t) \geq Kt$ for some constant K. Using Proposition 3 then gives that we converge for all t such that

$$t \geq \max\left\{\frac{2}{K}, \frac{1}{K}\frac{1}{\xi_i}\log\frac{1}{\delta(e^{\xi_i} - 1)}\right\} = \Theta\left(\log\frac{1}{\delta}\right) \tag{23}$$

(where $\xi_i = \frac{1}{4c_i + 2/3}$ as defined in Proposition 3).

When the system approaches consensus $X(t)$ will be sub-linear as $\mu_i(t) \to 0$ as $t \to \infty$; however, by analyzing the rate of convergence to consensus, we will obtain a better bound on $\mu_i(t)$ than $\Theta(1/t)$, which will yield a more precise convergence rate for the inherent belief estimator.

For the consensus case, Figure 3 shows an experiment approximating δ for given t^*.

VI. Bounds on Rate of Convergence to Consensus

In this section, we look at what the rate of convergence to consensus is, i.e. how quickly the time-averaged declared opinions $\beta_i(t)$ for each agent i converges to 0 or 1 in the case of consensus. The ultimate goal is to be able to characterize the probability of error of inherent belief estimator (7) using Proposition 3, illustrating how quickly we should expect the estimation of inherent beliefs to be correct.

Consensus occurs when either $\lambda_{\max}(\boldsymbol{J_0})$ or $\lambda_{\max}(\boldsymbol{J_1})$ is less than 1 (see [4]) . For this section, WLOG suppose that $\lambda_{\max}(\boldsymbol{J_0}) < 1$ meaning that consensus to $\mathbf{0}$ occurs. We will show that in the case of consensus, the convergence rate

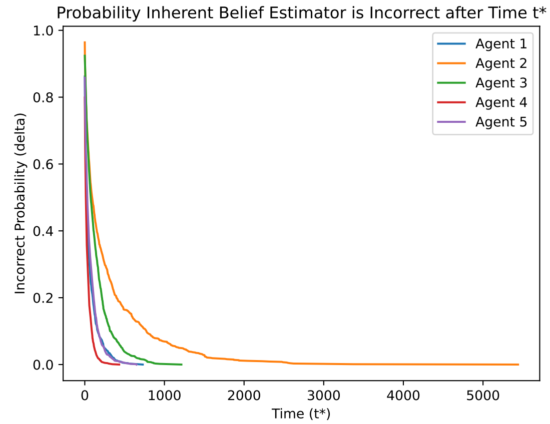

Fig. 3. Empirical plot (over 1000 experiments), for each agent i of the network in Figure 1, of the probability $\delta_i(t^*)$ that the MLE inherent belief estimator (7) is wrong for some $t \geq t^*$. Each experiment was run for 100000 steps; the curves are cut off at the first t^* such that (7) was always correct for all $t^* \leq t \leq 100000$ (i.e. $\delta = 0$ empirically).

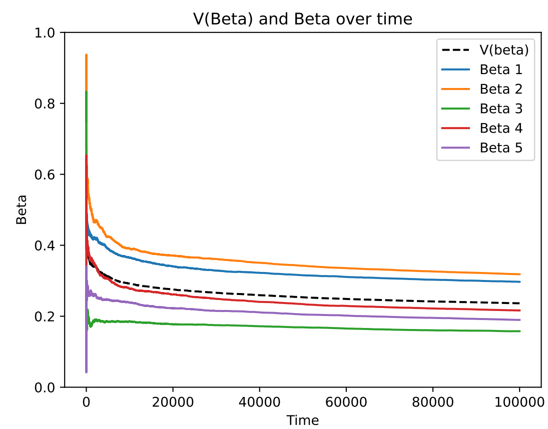

Fig. 4. Plot illustrating example evolution of $V(\boldsymbol{\beta}(t))$ and $\beta_i(t)$ for each agent i on the network in Figure 1.

of $\beta_i(t)$ and the estimator (7) is a function of $\lambda_{\max}(\boldsymbol{J_0})$. This eigenvalue is how the network structure influences these quantities.

We define that $\beta_i(t)$ for agent i converges to 0 at a rate of t^r if for every constant $\epsilon > 0$, we have that

$$\lim_{t\to\infty} \frac{\beta_i(t)}{t^{r+\epsilon}} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\beta_i(t)}{t^{r-\epsilon}} = \infty\,.$$

(We note that this definition does exclude subpolynomial terms, for instance, $t^r \log t$ and $t^r/\log t$ will both satisfy the conditions.) The goal is to show that $r = \lambda_{\max}(\boldsymbol{J_0}) - 1$.

In this section, first, we define a function V which is a scalar function of $\boldsymbol{\beta}$. We define linear processes h which can be used to upper and lower bound $V(\boldsymbol{\beta}(t))$. We then normalize process h to show the rate of convergence of $V(\boldsymbol{\beta}(t))$. Afterwards, we show that the convergence rate of each $\beta_i(t)$ is a constant multiplied by the convergence rate $V(\boldsymbol{\beta}(t))$. Finally, using the convergence rate of $\beta_i(t)$ with Proposition 3 gives a bound on the convergence rate of inherent belief estimate (7). An illustration of the decay rate of $\beta_i(t)$ and of $V(\boldsymbol{\beta}(t))$ for a given experiment is shown in Figure 4.

A. Definition of V and Linearized Processes

Let $\lambda = \lambda_{\max}(\boldsymbol{J_0})$. Let $\boldsymbol{v}$ be the associated (left) eigenvector of eigenvalue λ. Since $\boldsymbol{J_0}$ is irreducible (since $\boldsymbol{W}$ is irreducible) and a nonnegative matrix, the Perron-Frobenius theorem [21] implies that $\boldsymbol{v}$ is a positive eigenvector. Scale $\boldsymbol{v}$ so that $\boldsymbol{v}^\top \mathbf{1} = 1$. Let

$$V(\boldsymbol{\beta}) = \boldsymbol{v}^\top \boldsymbol{\beta}\,.$$

For convenience, we represent the declared opinions at a given step t as a (random) vector $\boldsymbol{\psi}(\boldsymbol{\beta}(t))$ depending on the state $\boldsymbol{\beta}(t)$. Thus the dynamics follow the update

$$\boldsymbol{\beta}(t+1) = \frac{t}{t+1}\boldsymbol{\beta}(t) + \frac{1}{t+1}\boldsymbol{\psi}(\boldsymbol{\beta}(t))$$

and $\boldsymbol{\psi}(\boldsymbol{\beta}(t))$ is a vector with ith component given by

$$\psi_i(\boldsymbol{\beta}(t)) = \begin{cases} 1 & \text{w.p. } f(\mu_i(t), \gamma_i) \\ 0 & \text{w.p. } 1 - f(\mu_i(t), \gamma_i) \end{cases}.$$

Since consensus to $\mathbf{0}$ occurs, this implies that $\lim_{t\to\infty} V(\boldsymbol{\beta}(t)) = 0$. We will show that the rate at which $V(\boldsymbol{\beta}(t))$ approaches 0 is also the rate at which each $\beta_i(t) \to 0$ up to a constant factor.

To analyze the convergence of $V(\boldsymbol{\beta}(t))$, we will define linear functions which we can use to upper and lower bound the value of $V(\boldsymbol{\beta}(t)$ when $\boldsymbol{\beta}(t)$ is in a certain region.

For $\zeta > 0$, let random vector $\bar{\boldsymbol{\psi}}(\boldsymbol{\beta}(t), \zeta)$ be such that the ith component is given by

$$\bar{\psi}_i(\boldsymbol{\beta}(t), \zeta) = \begin{cases} \begin{cases} 1 & \text{w.p. } \zeta\gamma_i\mu_i(t) \\ 0 & \text{w.p. } 1 - \zeta\gamma_i\mu_i(t) \end{cases} & \text{if } \zeta\gamma_i\mu_i(t) \le 1 \\ \zeta\gamma_i\mu_i(t) & \text{if } \zeta\gamma_i\mu_i(t) > 1 \end{cases} \tag{24}$$

(The case where $\zeta\gamma_i\mu_i(t) > 1$ is a technicality we need to consider) where $\bar{\boldsymbol{\psi}}(\boldsymbol{\beta}(t), \zeta)$ is also maximally coupled with $\boldsymbol{\psi}(\boldsymbol{\beta}(t))$. This means that the joint distribution between $\bar{\boldsymbol{\psi}}(\boldsymbol{\beta}(t), \zeta)$ and $\boldsymbol{\psi}(\boldsymbol{\beta}(t))$ is such that

$$\mathbb{P}[\bar{\psi}_i(\boldsymbol{\beta}(t), \zeta) = \psi_i(\boldsymbol{\beta}(t))]$$

is maximized. Note that by (24),

$$\begin{aligned} \mathbb{E}[\bar{\psi}_i(\boldsymbol{\beta}(t), \zeta)\,|\,\mu_i(t)] &= \zeta\gamma_i\mu_i(t) \quad \text{and} \\ \mathrm{Var}[\bar{\psi}_i(\boldsymbol{\beta}(t), \zeta)\,|\,\mu_i(t)] &\le \zeta\gamma_i\mu_i(t)\,. \end{aligned} \tag{25}$$

where the variance term follows from

$$\mathrm{Var}[\bar{\psi}_i(\boldsymbol{\beta}(t), \zeta)\,|\,\mu_i(t)] = \max(\zeta\gamma_i\mu_i(t)(1 - \zeta\gamma_i\mu_i(t)), 0)\,.$$

Definition 5 (Linearized Process)**.** *Given t_0 and $\boldsymbol{\beta}(t_0)$, for each constant value $\alpha > 0$ define the stochastic process $h^\alpha(t)$ for $t \ge t_0$ to have the following values which determine its joint distribution with random stochastic process $\boldsymbol{\beta}(t)$:*

$$h^\alpha(t_0) = V(\boldsymbol{\beta}(t_0))$$
$$h^\alpha(t+1) = \frac{t}{t+1}h^\alpha(t) + \frac{1}{t+1}\boldsymbol{v}^\top \bar{\boldsymbol{\psi}}\left(\boldsymbol{\beta}(t), \alpha\frac{h^\alpha(t)}{V(\boldsymbol{\beta}(t))}\right).$$

While in the above $h^\alpha(t)$ is defined with respect to random process $\boldsymbol{\beta}(t)$, the marginal distribution of $h^\alpha(t)$ will have expectation and variance bounds that does not depend on $\boldsymbol{\beta}(t)$. The processes $h^\alpha(t)$ will be used as linear upper and lower bounds to $V(\boldsymbol{\beta}(t))$. Let us define

$$R(t+1, c) = \prod_{\tau=0}^{t} \frac{\tau + c}{\tau + 1}$$

for $c \in (0,1)$ and time t. We then get the following result:

Lemma 6. *Let the gamma function be represented by* $\Gamma(\cdot)$.

$$\frac{1}{\Gamma(c)(t+1)^{1-c}} \le R(t,c) \le \frac{1}{\Gamma(c)(t)^{1-c}}$$

Proof. First,

$$R(t,c) = \frac{\Gamma(t+c)}{\Gamma(c)\Gamma(t+1)}\,.$$

Since $c \in (0,1)$, using Gautschi's inequality gives that

$$\frac{1}{(t+1)^{1-c}} \le \frac{\Gamma(t+c)}{\Gamma(t+1)} \le \frac{1}{t^{1-c}}$$

Including the $\Gamma(c)$ term in the denominator gives the result. □

Lemma 7. *Some properties of $h^\alpha(\cdot)$ are:*

(a) For any $\epsilon > 0$, there is a time t_0, where for $t > t_0$, $\boldsymbol{\beta}(t)$ remains in a disc of radius $r(\epsilon)$ from $\mathbf{0}$*. Then the trajectory $V(\boldsymbol{\beta}(t))$ is such that there exists an $1 < \alpha_+ < 1 + \epsilon$ and $1 - \epsilon < \alpha_- < 1$ such that*

$$h^{\alpha_-}(t) \le V(\boldsymbol{\beta}(t)) \le h^{\alpha_+}(t)$$

for $t > t_0$.

(b) For $t \ge t_1$ and any $\alpha > 0$ where $\alpha\lambda < 1$,

$$\begin{aligned} \mathbb{E}[h^\alpha(t)|h^\alpha(t_1)] &= \frac{R(t, \alpha\lambda)}{R(t_1, \alpha\lambda)}h(t_1) \\ &= (1 + o(1))\frac{t^{\alpha\lambda-1}}{t_1^{\alpha\lambda-1}}h(t_1) \end{aligned}$$

(c) For sufficiently large t_1 there is a constant c where for $t > t_1$ and any $\alpha > 0$,

$$\mathrm{Var}[h^\alpha(t)|h^\alpha(t_1)] \le c\frac{t^{2\alpha\lambda-2}}{t_1^{2\alpha\lambda-1}}h^\alpha(t_1)$$

Proof. Proof of Part (a):

From [4, Theorem 2], we know that $\boldsymbol{\beta}(t)$ almost surely converges to an equilibrium point. In this case, that equilibrium point is $\mathbf{0}$ and thus there must be some time t_0 after which which $\boldsymbol{\beta}(t)$ lies in a disc around $\mathbf{0}$. When $\boldsymbol{\beta}(t)$ is within a disc of radius r of $\mathbf{0}$, each $\mu_i(t)$ must also be within r of 0.

There are constants α_- and α_+ so that for each agent i,

$$\alpha_-\gamma_i\mu_i \le f(\mu_i, \gamma_i) \le \alpha_+\gamma_i\mu_i$$

for all $\mu_i \le r$. (The values α_- and α_+ get closer to 1 as r decreases.) We focus on showing $h^{\alpha_-}(t) \le V(\boldsymbol{\beta}(t))$ (showing the other case is symmetric).

First, by definition $h^{\alpha_-}(t_0) = V(\boldsymbol{\beta}(t_0))$. If we assume that $h^{\alpha_-}(t) \le V(\boldsymbol{\beta}(t))$, then

$$\begin{aligned} &h^{\alpha_-}(t+1) \\ &= \frac{t}{t+1} h^{\alpha_-}(t) + \frac{1}{t+1} \boldsymbol{v}^\top \bar{\boldsymbol{\psi}} \left(\boldsymbol{\beta}(t), \alpha_- \frac{h^{\alpha_-}(t)}{V(\boldsymbol{\beta}(t))} \right) \\ &\le \frac{t}{t+1} V(\boldsymbol{\beta}(t)) + \frac{1}{t+1} \boldsymbol{v}^\top \bar{\boldsymbol{\psi}} \left(\boldsymbol{\beta}(t), \alpha_- \frac{h^{\alpha_-}(t)}{V(\boldsymbol{\beta}(t))} \right) \end{aligned}$$

Since $h^{\alpha_-}(t) \le V(\boldsymbol{\beta}(t))$, random variable

$$\bar{\psi}_i \left(\boldsymbol{\beta}(t), \alpha_- \frac{h^{\alpha_-}(t)}{V(\boldsymbol{\beta}(t))} \right) \tag{26}$$

is 1 with probability

$$\alpha_- \frac{h^{\alpha_-}(t)}{V(\boldsymbol{\beta}(t))} \gamma_i \mu_i(t) \le \alpha_- \gamma_i \mu_i(t) \le f(\mu_i, \gamma_i)$$

where $f(\mu_i, \gamma_i)$ is the probability of $\psi_i(\boldsymbol{\beta}(t)) = 1$. Because the two variables are maximally coupled, every instance when (26) is 1, it must also be that $\psi_i(\boldsymbol{\beta}(t)) = 1$. Then

$$\bar{\psi}_i \left(\boldsymbol{\beta}(t), \alpha_- \frac{h^{\alpha_-}(t)}{V(\boldsymbol{\beta}(t))} \right) \le \psi_i(\boldsymbol{\beta}(t))$$

and thus

$$\begin{aligned} h^{\alpha_-}(t+1) &\le \frac{t}{t+1} V(\boldsymbol{\beta}(t)) + \frac{1}{t+1} \boldsymbol{v}^\top \boldsymbol{\psi}(\boldsymbol{\beta}(t)) \\ &= V(\boldsymbol{\beta}(t+1)) \,. \end{aligned}$$

Proof of Part (b): We will use proof by induction. For the base case we trivially have

$$\mathbb{E}[h^\alpha(t_1) | h^\alpha(t_1)] = h^\alpha(t_1) = \frac{R(t_1, \alpha\lambda)}{R(t_1, \alpha\lambda)} h^\alpha(t_1) \,.$$

For the inductive step, we can assume that

$$\mathbb{E}[h^\alpha(t) | h^\alpha(t_1)] = \frac{R(t, \alpha\lambda)}{R(t_1, \alpha\lambda)} h^\alpha(t_1) \,.$$

Then we consider $t+1$:

$$\begin{aligned} &\mathbb{E}[h^\alpha(t+1) | h^\alpha(t_1)] = \frac{t}{t+1} \mathbb{E}[h^a(t) | h^\alpha(t_1)] \\ &\quad + \frac{1}{t+1} \mathbb{E}\left[\boldsymbol{v}^\top \bar{\boldsymbol{\psi}} \left(\boldsymbol{\beta}(t), \alpha \frac{h^\alpha(t)}{V(\boldsymbol{\beta}(t))} \right) \middle| h^\alpha(t_1) \right] \\ &= \frac{t}{t+1} \mathbb{E}[h^a(t) | h^\alpha(t_1)] \\ &\quad + \frac{1}{t+1} \mathbb{E}\left[\boldsymbol{v}^\top \frac{h^\alpha(t)}{V(\boldsymbol{\beta}(t))} \alpha \boldsymbol{\Gamma} \boldsymbol{W} \boldsymbol{\beta}(t) \middle| h^\alpha(t_1) \right] \\ &= \frac{t}{t+1} \mathbb{E}[h^a(t) | h^\alpha(t_1)] \\ &\quad + \frac{1}{t+1} \mathbb{E}\left[\frac{h^\alpha(t)}{V(\boldsymbol{\beta}(t))} \alpha\lambda \boldsymbol{v}^\top \boldsymbol{\beta}(t) \middle| h^\alpha(t_1) \right] \\ &= \frac{t}{t+1} \mathbb{E}[h^\alpha(t) | h^\alpha(t_1)] + \frac{1}{t+1} \alpha\lambda \mathbb{E}[h^\alpha(t) | h^\alpha(t_1)] \\ &= \frac{R(t+1, \alpha\lambda)}{R(t_1, \alpha\lambda)} h(t_1) \end{aligned}$$

Then by Lemma 6 , we have

$$\frac{(t+1)^{\alpha\lambda-1}}{t_1^{\alpha\lambda-1}} \le \frac{R(t, \alpha\lambda)}{R(t_1, \alpha\lambda)} \le \frac{t^{\alpha\lambda-1}}{(t_1+1)^{\alpha\lambda-1}}$$

Thus, for any $\epsilon > 0$, there is a sufficiently large t_1 such that for all $t \ge t_1$,

$$(1-\epsilon) \frac{t^{\lambda-1}}{t_1^{\lambda-1}} h^\alpha(t_1) \le \mathbb{E}[h^\alpha(t) \mid h^\alpha(t_1)] \le (1+\epsilon) \frac{t^{\lambda-1}}{t_1^{\lambda-1}} h^\alpha(t_1) \,.$$

Proof of Part (c):

When we apply the law of total variance, we get that

$$\begin{aligned} &\mathrm{Var}[h^\alpha(t+1) | h^\alpha(t_1)] \\ &\quad = \mathbb{E}\left[\mathrm{Var}[h^\alpha(t+1) | \mathcal{H}_t, h^\alpha(t_1)] \middle| h^\alpha(t_1) \right] \\ &\qquad + \mathrm{Var}\left[\mathbb{E}[h^\alpha(t+1) | \mathcal{H}_t, h^\alpha(t_1)] \middle| h^\alpha(t_1) \right] \,. \end{aligned} \tag{27}$$

We compute the first term in (27) by starting with

$$\begin{aligned} &\mathrm{Var}\left[\bar{\psi}_i \left(\boldsymbol{\beta}(t), \alpha \frac{h^\alpha(t)}{V(\boldsymbol{\beta}(t))} \right) \middle| \mathcal{H}_t, h^\alpha(t_1) \right] \\ &\qquad \le \alpha \frac{h^\alpha(t)}{V(\boldsymbol{\beta}(t))} \gamma_i \mu_i(t) \,. \end{aligned}$$

where we used (25).

Let $c_0 = \max_i \{v_i\}$. Then we can compute

$$\begin{aligned} &\mathrm{Var}[h^\alpha(t+1) | \mathcal{H}_t, h^\alpha(t_1)] \\ &= \mathrm{Var}\left[\frac{1}{t+1} \boldsymbol{v}^\top \bar{\boldsymbol{\psi}} \left(\boldsymbol{\beta}(t), \alpha \frac{h^\alpha(t)}{V(\boldsymbol{\beta}(t))} \right) \middle| \mathcal{H}_t, h^\alpha(t_1) \right] \\ &= \frac{1}{(t+1)^2} \sum_{i=1}^n v_i^2 \mathrm{Var}\left[\bar{\psi}_i \left(\boldsymbol{\beta}(t), \alpha \frac{h^\alpha(t)}{V(\boldsymbol{\beta}(t))} \right) \middle| \mathcal{H}_t, h^\alpha(t_1) \right] \\ &\le \frac{1}{(t+1)^2} \sum_{i=1}^n v_i^2 \alpha \frac{h^\alpha(t)}{V(\boldsymbol{\beta}(t))} \gamma_i \mu_i(t) \\ &\le \frac{1}{(t+1)^2} \alpha \frac{h^\alpha(t)}{V(\boldsymbol{\beta}(t))} c_0 \lambda \boldsymbol{v}^\top \boldsymbol{\beta}(t) = \frac{c_0 \alpha\lambda}{(t+1)^2} h^\alpha(t) \,. \end{aligned}$$

To finish computing the first term in (27), we have

$$\begin{aligned} &\mathbb{E}\left[\mathrm{Var}[h^\alpha(t+1) | \mathcal{H}_t, h^\alpha(t_1)] \middle| h^\alpha(t_1) \right] \\ &\quad \le \mathbb{E}\left[\frac{c_0 \alpha\lambda}{(t+1)^2} h^\alpha(t_1) \middle| h^\alpha(t_1) \right] \\ &\quad \le c_1 \alpha\lambda \frac{1}{t^2} \frac{t^{\alpha\lambda-1}}{t_1^{\alpha\lambda-1}} h^\alpha(t_1) \end{aligned}$$

where we used the approximation result from Lemma 7(b) which is a correct upper bound for some constant c_1 and sufficiently large t_1. For the second term in (27), the expectation inside the variance is

$$\mathbb{E}[h^\alpha(t+1) | \mathcal{H}_t, h^\alpha(t_1)] = \frac{R(t+1, \alpha\lambda)}{R(t, \alpha\lambda)} h^\alpha(t)$$

and thus

$$\begin{aligned} &\mathrm{Var}\left[\mathbb{E}[h^\alpha(t+1) | \mathcal{H}_t, h^\alpha(t_1)] \middle| h^\alpha(t_1) \right] \\ &\qquad = \left(\frac{R(t+1, \alpha\lambda)}{R(t, \alpha\lambda)} \right)^2 \mathrm{Var}[h^\alpha(t) | h^\alpha(t_1)] \,. \end{aligned}$$

Putting this together gets

$$\begin{aligned} \mathrm{Var}[h^\alpha(t+1) | h^\alpha(t_1)] &\le c_1 \alpha\lambda \frac{1}{t^2} \frac{t^{\alpha\lambda-1}}{t_1^{\alpha\lambda-1}} h^\alpha(t_1) \\ &\quad + \left(\frac{R(t+1, \alpha\lambda)}{R(t, \alpha\lambda)} \right)^2 \mathrm{Var}[h^\alpha(t) | h^\alpha(t_1)] \,. \end{aligned}$$

Telescoping the variance terms yields:

$$\begin{aligned}
&\mathrm{Var}[h^\alpha(t+1)|h^\alpha(t_1)] \\
&\quad \le c_2\alpha\lambda \sum_{\tau=t_1}^{t}\left(\frac{(t+1)^{\alpha\lambda-1}}{\tau^{\alpha\lambda-1}}\right)^2 \frac{1}{\tau^2}\frac{\tau^{\alpha\lambda-1}}{t_1^{\alpha\lambda-1}} h^\alpha(t_1) \\
&\quad = c_2\alpha\lambda \frac{(t+1)^{2\alpha\lambda-2}}{t_1^{\alpha\lambda-1}} h^\alpha(t_1) \sum_{\tau=t_1}^{t} \frac{1}{\tau^{\alpha\lambda+1}} .
\end{aligned}$$

Approximating the sum using an integral then gives:

$$\sum_{\tau=t_1}^{t} \frac{1}{\tau^{\alpha\lambda+1}} \le \int_{t_1}^{\infty} \frac{1}{\tau^{\alpha\lambda+1}} d\tau = \frac{1}{\alpha\lambda} t_1^{-\alpha\lambda}$$

which results in

$$\begin{aligned}
\mathrm{Var}[h^\alpha(t)|h^\alpha(t_1)] &\le c_2\alpha\lambda \frac{t^{2\alpha(\lambda-1)}}{t_1^{\alpha\lambda-1}} h^\alpha(t_1) \frac{1}{\alpha\lambda} t_1^{-\alpha\lambda} \\
&= c_2 \frac{t^{2(\alpha\lambda-1)}}{t_1^{\alpha\lambda-1}} h^\alpha(t_1) t_1^{-\alpha\lambda} .
\end{aligned}$$

where c_2 is a constant not depending on α, λ, or t (so long as t_1 is sufficiently large). □

B. Normalized Convergence Process

Using the linearized process, we will define a normalized process whose asymptotic convergence properties will be used to show our desired results. We will first use the results from Lemma 7 to get a bound on the variance of the linearized process in terms of the square of its expected value. For this bound, we need to use that at any time t, we expect that $V(\boldsymbol{\beta}(t)) > c\kappa/t \triangleq \kappa^*/t$ (this is discussed in (IV-A1)).

Lemma 8. *For any sufficiently large t_0 and any $t > t_0$,*

$$\frac{\mathrm{Var}[h^\alpha(t)\,|\,h^\alpha(t_0)]}{\mathbb{E}[h^\alpha(t)\,|\,h^\alpha(t_0)]^2} \le \frac{c}{h^\alpha(t_0)t_0} \le c^*$$

for some constant c^* which does not depend on t or t_0.

Proof.

$$\begin{aligned}
\frac{\mathrm{Var}[h^\alpha(t)\,|\,h^\alpha(t_0)]}{\mathbb{E}[h^\alpha(t)\,|\,h^\alpha(t_0)]^2} &\le \frac{c\frac{t^{(\alpha\lambda-1)2}}{t_0^{2\alpha\lambda-1}} h^\alpha(t_0)}{\left(\frac{t^{\alpha\lambda-1}}{t_0^{\alpha\lambda-1}} h^\alpha(t_0)\right)^2} \\
&= \frac{c\, t_0^{\alpha\lambda-1}}{h^\alpha(t_0)} \frac{1}{t_0^{\alpha\lambda}} = \frac{c}{h^\alpha(t_0)t_0} \le c^*
\end{aligned}$$

where the last step uses $h^\alpha(t_0) = V(\boldsymbol{\beta}(t_0)) = \Omega(t_0^{-1})$. □

Note that $h^\alpha(t_0) \propto t_0^{-1}$ is a (guaranteed, not probabilistic) worst-case bound, and significantly worse than the expected $h^\alpha(t_0) \propto t_0^{\alpha\lambda-1}$. Note also that replacing $h^\alpha(t)$ on the left hand side by a scaled version $\rho\, h^\alpha(t)$ (where ρ can depend on t, t_0 but not on the value of $h^\alpha(t)$ itself) will not change the bound, as it multiplies both the numerator and denominator by ρ^2. We therefore define a martingale $\bar{h}^\alpha(t)$ as follows:

Definition 6. *Given a t_0, consider the process $\bar{h}^\alpha(\cdot)$ starting from time t_0: $\bar{h}^\alpha(t_0) = h^\alpha(t_0)$; then for any $t > t_0$ we define the normalized convergence process as*

$$\bar{h}^\alpha(t) = \frac{R(t_0, \alpha\lambda)}{R(t, \alpha\lambda)} h^\alpha(t) .$$

We also define the random variable $\bar{h}_\alpha$ as follows:

$$\bar{h}_\alpha = \liminf_{t\to\infty} \bar{h}^\alpha(t) .$$

Then $\bar{h}^\alpha(t)$ is nonnegative, uniformly integrable, and is a martingale.

Lemma 9. *The sequence $\{\bar{h}^\alpha(t)\}_{t\ge t_0}$ is a uniformly integrable martingale and $\lim_{t\to\infty} \bar{h}^\alpha(t) = \bar{h}_\alpha$ almost surely. Furthermore, for any t, we have $\bar{h}^\alpha(t) = \mathbb{E}[\bar{h}_\alpha\,|\,\mathcal{H}(t)]$.*

Proof. The process $\{\bar{h}^\alpha(t)\}_{t\ge t_0}$ is a martingale due to Lemma 7(b). The Martingale Convergence Theorem shows that it converges to a well-defined (random) limit almost surely, since $\bar{h}^\alpha(t) \ge 0$ and by definition

$$\mathbb{E}[|\bar{h}^\alpha(t)|] = \mathbb{E}[\bar{h}^\alpha(t)] = \bar{h}^\alpha(t_0) < \infty$$

Note that this means that almost surely,

$$\bar{h}_\alpha = \lim_{t\to\infty} \bar{h}^\alpha(t)$$

as the limit almost surely exists.

Finally, Lemma 8 (and the fact that $\{\bar{h}^\alpha(t)\}_{t\ge t_0}$ is a martingale) shows that $\mathbb{E}[|\bar{h}^\alpha(t)|^2]$ is bounded for all t, which implies uniform integrability by [22, Section 13.3]. □

Then Lemma 8 yields the following:

Corollary 1. *For any sufficiently large t^*, for any $t > t^*$*

$$\mathbb{P}\left[\bar{h}^\alpha(t) \le \frac{\bar{h}^\alpha(t^*)}{2} \,\middle|\, \mathcal{H}(t^*)\right] \le \frac{c^*}{c^* + 1/4}$$

This also implies that for any sufficiently large t^*,

$$\mathbb{P}\left[\bar{h}_\alpha \le \frac{\bar{h}^\alpha(t^*)}{4} \,\middle|\, \mathcal{H}(t^*)\right] \le \frac{c^*}{c^* + 1/8} \tag{28}$$

where c^* is the constant used in Lemma 8.

Proof. Note that if $\bar{h}^\alpha(t) \le \frac{\bar{h}^\alpha(t^*)}{2}$ then $\bar{h}^\alpha(t^*) - \bar{h}^\alpha(t) \ge \frac{\bar{h}^\alpha(t^*)}{2}$; since $\bar{h}^\alpha(t^*) - \bar{h}^\alpha(t)$ has mean 0 (conditioned on $\mathcal{H}(t^*)$) and variance $\le c^*$ given by Lemma 8, the Chebyshev-Cantelli inequality states that

$$\begin{aligned}
&\mathbb{P}\left[\bar{h}^\alpha(t^*) - \bar{h}^\alpha(t) \ge \frac{\bar{h}^\alpha(t^*)}{2} \middle| \mathcal{H}(t^*)\right] \\
&\quad \le \frac{\mathrm{Var}[\bar{h}^\alpha(t)]}{\mathrm{Var}[\bar{h}^\alpha(t)] + (\frac{\bar{h}^\alpha(t^*)}{2})^2} \\
&\quad \le \frac{c^*\bar{h}^\alpha(t^*)^2}{c^*\bar{h}^\alpha(t^*)^2 + (\frac{\bar{h}^\alpha(t^*)}{2})^2} = \frac{c^*}{c^* + 1/4}
\end{aligned}$$

Note that the function $\frac{x}{x+y}$ is increasing in x if y is positive, so we appropriately get an upperbound when applying $\mathrm{Var}[\bar{h}^\alpha(t)] \le c^*\bar{h}^\alpha(t^*)^2$.

To prove (28) (note that the bound is now $\frac{\bar{h}^\alpha(t^*)}{4}$ rather than $\frac{\bar{h}^\alpha(t^*)}{2}$) we have that by Lemma 9, $\bar{h}^\alpha(t) \to \bar{h}_\alpha$ almost surely; this means that $\bar{h}^\alpha(t) \to \bar{h}_\alpha$ in probability as well, so for any $\delta_1, \delta_2 > 0$, and for sufficiently large t,

$$\mathbb{P}[|\bar{h}_\alpha - \bar{h}^\alpha(t)| > \delta_1] \le \delta_2 \tag{29}$$

Set $\delta_1 = \frac{\bar{h}^\alpha(t^*)}{4}$ and $\delta_2 = \frac{c^*}{c^*+1/8} - \frac{c^*}{c^*+1/4}$. Then we assume to the contrary that

$$\mathbb{P}\left[\bar{h}_\alpha \leq \frac{\bar{h}^\alpha(t^*)}{4} \,\middle|\, \mathcal{H}(t^*)\right] > \frac{c^*}{c^*+1/8}$$

This means that, for any sufficiently large t,

$$\mathbb{P}\left[\bar{h}_\alpha \leq \frac{\bar{h}^\alpha(t^*)}{4} \text{ and } \bar{h}^\alpha(t) > \frac{\bar{h}^\alpha(t^*)}{2} \,\middle|\, \mathcal{H}(t^*)\right] > \frac{c^*}{c^*+1/8} - \frac{c^*}{c^*+1/4}$$

which yields the desired contradiction given by (29). □

Lemma 10. *For any α, almost surely $\bar{h}_\alpha > 0$*

Proof. Therefore we have established that for any α:

- $\bar{h}^\alpha(t)$ is a uniformly integrable martingale;
- $\bar{h}^\alpha(t) \to \bar{h}_\alpha$ almost surely;
- for any sufficiently large t^* (28) holds which implies that for all sufficiently large t^*,

$$\mathbb{P}[\bar{h}_\alpha > 0 \,|\, \mathcal{H}(t^*)] \geq 1 - \frac{c^*}{c^*+1/8} > 0$$

We define the process $\eta(t)$ as

$$\eta(t) = \mathbb{P}[\bar{h}_\alpha > 0 \,|\, \mathcal{H}(t)]$$

which is a martingale due to the tower property (and uniformly integrable because it is bounded). We likewise define

$$\eta = \mathbf{1}\{\bar{h}_\alpha > 0\}$$

and (due to uniform integrability) almost surely $\eta(t) \to \eta$ because of Levy's 0-1 Law [23, Theorem 5.5.8]; specifically, since η is an indicator function and is fully determined by the filtration $\bigcup_t \mathcal{H}(t)$ we know by Levy's Upward Theorem (applied to $\eta(t)$) that

$$\lim_{t\to\infty} \eta(t) = \lim_{t\to\infty} \mathbb{P}[\bar{h}_\alpha > 0 \,|\, \mathcal{H}(t)] = \lim_{t\to\infty} \mathbb{E}[\eta \,|\, \mathcal{H}(t)] \stackrel{a.s.}{=} \mathbb{E}\left[\eta \,\middle|\, \bigcup_t \mathcal{H}(t)\right] = \eta \in \{0,1\}\,.$$

But $\lim_{t\to\infty} \eta(t) \geq 1 - \frac{c^*}{c^*+1/8} > 0$, thus showing that 1 is the only possible limit out of $\{0,1\}$. Thus, $\eta = 1$ almost surely, so $\bar{h}_\alpha > 0$ almost surely. □

C. Rate of Decay for V and β_i

Theorem 5. *For any $\epsilon > 0$, we have that*

$$\lim_{t\to\infty} \frac{V(\boldsymbol{\beta}(t))}{t^{\lambda-1+\epsilon}} = 0 \text{ and } \lim_{t\to\infty} \frac{V(\boldsymbol{\beta}(t))}{t^{\lambda-1-\epsilon}} = \infty$$

almost surely.

Proof. For a given ϵ, using Lemma 7(a), we can find a $\delta(\epsilon)$ with corresponding t_0 large enough and an α_+ and α_- which are such that

$$\alpha_+ < 1 + \epsilon/\lambda \quad\text{and}\quad \alpha_- > 1 - \epsilon/\lambda$$

Then for any trajectory $V(\boldsymbol{\beta}(t))$, there exists some trajectory $h^{\alpha_-}(t)$ and $h^{\alpha_+}(t)$ such that

$$h^{\alpha_-}(t) \leq V(\boldsymbol{\beta}(t)) \leq h^{\alpha_+}(t)$$

Using Lemma 10, any trajectory of $h^{\alpha_-}(t)$ has a corresponding martingale $\bar{h}^{\alpha_-}(t)$ which converges to a constant. Thus $h^{\alpha_-}(t)$ converges to zero at a rate of $\Omega(t^{\alpha_- \lambda - 1})$ almost surely. Similarly, $h^{\alpha_+}(t)$ converges to zero at a rate of $\Omega(t^{\alpha_+ \lambda - 1})$ almost surely.

Since the value of t_0 does not affect the asymptotic rate after t_0, we have

$$\lim_{t\to\infty} \frac{V(\boldsymbol{\beta}(t))}{t^{\lambda-1+\epsilon}} \leq \lim_{t\to\infty} \frac{h^{\alpha_+}(t)}{t^{\lambda-1+\epsilon}} \leq \lim_{t\to\infty} \frac{c_+ t^{\alpha_+\lambda-1}}{t^{\lambda-1+\epsilon}} = 0$$
$$\lim_{t\to\infty} \frac{V(\boldsymbol{\beta}(t))}{t^{\lambda-1-\epsilon}} \geq \lim_{t\to\infty} \frac{h^{\alpha_-}(t)}{t^{\lambda-1-\epsilon}} \geq \lim_{t\to\infty} \frac{c_- t^{\alpha_-\lambda-1}}{t^{\lambda-1-\epsilon}} = \infty\,.$$

□

Next we need the above result on $V(\boldsymbol{\beta}(t))$ to imply a result on all $\beta_i(t)$.

Lemma 11. *Given any $\epsilon > 0$, there exists some time t_0 and some c, such that for all $t > t_0$, there exists some i such that*

$$\beta_i(t) \geq c t^{\lambda-1-\epsilon} \tag{30}$$

almost surely. Additionally, for any $t_2 > t_0$, there exists some i such that $\beta_i(t) \geq c t^{\lambda-1-\epsilon}$ holds at least $1/n$ of the times t in $t_2 < t \leq 2t_2$.

Note that at this point in the development of our results, the lemma does not guarantee that the same i will satisfy the property (30). Hence why we have the second statement saying that there exists some i that satisfies (30) some fraction of the time.

Proof. First, $V(\boldsymbol{\beta}(t)$ converges at a rate at least $\Omega(t^{\lambda-1-\epsilon})$. Thus, there is some t_0, where for $t > t_0$,

$$V(\boldsymbol{\beta}(t)) \geq c t^{\lambda-1-\epsilon}\,.$$

At time $t > t_0$, let k be such that $\beta_k(t) \geq \beta_i(t)$ for all i. Since $\boldsymbol{v}$ is a vector so that $\boldsymbol{v}^\top \mathbf{1} = 1$, we have that

$$\beta_k(t) = (\boldsymbol{v}^\top \mathbf{1})\beta_k(t) \geq \boldsymbol{v}^\top \boldsymbol{\beta}(t) = V(\boldsymbol{\beta}(t))\,.$$

And thus this shows (30) for each t.

For the second statement, we know that there must be some i which satisfies (30). Since there are only n candidates for i, at least one i must occur the most often, which means this i occurs at least $1/n$ of the time in a certain interval. □

We eventually want to show that $\beta_i(t) \geq c t^{\lambda-1-\epsilon}$ for some c for all t larger than some t^* for all i. To help show this, we will first look at a weaker case, when this equation holds for some portion of the time. This will help prove what we need.

Definition 7. *For any agent i and constants $c, \rho > 0$, we say that time $t_1 > 0$ is (c,ρ)-good for agent i if*

$$\beta_i(t) \geq c t^{\lambda-1-\epsilon}$$

for at least ρ fraction of the times $t \in [t_1/2, t_1]$. Let the set of times where this holds be

$$\mathcal{T}_i(c,\rho) = \{t_1 : t_1 \text{ is } (c,\rho)\text{-good for } i\}$$

For shorthand, once c, ρ are fixed, we denote the set of good times for i as $\mathcal{T}_i$. The key observation is that the set of good times for an agent i eventually becomes good for all her neighbors j, which then eventually become good for all their neighbors, and so forth until the set of good times for i must be good for all agents.

Lemma 12. *Fix an agent i and constants $c_i, \rho_i > 0$, and let $\mathcal{T}_i := \mathcal{T}_i(c_i, \rho_i)$, and let j be adjacent to i. Then there is some $c_j, \rho_j > 0$ such that, almost surely, there is some t^* for which*

$$t_1 > t^* \text{ and } t_1 \in \mathcal{T}_i \implies t_1 \in \mathcal{T}_j$$

where $\mathcal{T}_j := \mathcal{T}_j(c_j, \rho_j)$.

Note that ρ_j can be less than ρ_i, meaning that in the argument where good times spread from a source i, the ρ's diminish as the process gets further from i; however, since the graph is finite, it remains bounded away from 0 over the whole graph.

Proof. Recall that $t_1 \in \mathcal{T}_i$ means that $\beta_i \geq c_i t^{\lambda-1-\epsilon}$ for at least a ρ_i fraction of $t \in [t_1/2, t_1]$; we say a time t is c_i-*enough* (for agent i) if $\beta_i \geq c_i t^{\lambda-1-\epsilon}$ (unlike *good* times t_1, this only depends on the value of β_i at time t, not at any previous time).

First, note that at least $(\rho_i/4)t_1$ different t in $[t_1/2, (1-\rho_i/4)t_1]$ are c_i-enough (since there are at least $(\rho_i/2)t_1$ c_i-enough times in total). For any $t \in [t_1/2, t_1]$,

$$\beta_i(t) \geq \begin{cases} c_i t_1^{\lambda-1-\epsilon} & \text{if } t \text{ is } c_i\text{-enough} \\ 0 & \text{otherwise} \end{cases}$$

since $t \leq t_1$, and therefore (as $w_{j,i} = \frac{a_{j,i}}{\mathsf{deg}(j)}$)

$$\mu_j(t) \geq \begin{cases} c_i w_{j,i} t_1^{\lambda-1-\epsilon} & \text{if } t \text{ is } c_i\text{-enough} \\ 0 & \text{otherwise} \end{cases}.$$

So, when t is c_i-enough, we get

$$\begin{aligned} f(\mu_j(t), \gamma_j) &\geq f(w_{j,i} c_i t_1^{\lambda-1-\epsilon}, \gamma_j) \\ &= \frac{\gamma_j w_{j,i} c_i t_1^{\lambda-1-\epsilon}}{\gamma_j w_{j,i} c_i t_1^{\lambda-1-\epsilon} + 1 - w_{j,i} c_i t_1^{\lambda-1-\epsilon}} \\ &\geq c_j' t_1^{\lambda-1-\epsilon} \end{aligned}$$

where $c_j' = \min(\gamma_j, 1) w_{j,i} c_i$, which is a lower bound on the probability of agent j declaring 1 at any c_i-enough time. Consider the $\geq \rho_i/4$ such times in $[t_1/2, (1-\rho_i/4)t_1]$; the number of 1's declared by j in the range $[t_1/2, (1-\rho_i/4)t_1]$ thus stochastically dominates the sum of $(\rho_i/4)t_1$ independent Bernoulli random variables with probability $c_j' t_1^{\lambda-1-\epsilon}$ each (whose sum has expected value $(\rho_i/4)c_j' t_1^{\lambda-\epsilon}$). By the Chernoff bound, this then yields that

$$\mathbb{P}\left[\sum_{t=t_1/2}^{(1-\rho_i/4)t_1} \psi_{j,t} \leq \frac{\rho_i c_j'}{8} t_1^{\lambda-\epsilon}\right] \leq e^{-\frac{\rho_i}{32} c_j' t_1^{\lambda-\epsilon}} \tag{31}$$

meaning that there is a very high probability of getting at least $\frac{\rho_i c_j'}{8} t_1^{\lambda-\epsilon}$ declarations of 1 from agent j by time $(1-\rho_i/4)t_1$. But then for the $(\rho_i/4)t_1$ times $t \in [(1-\rho_i/4)t_1, t_1]$, we have

$$\beta_j(t) \geq \frac{\rho_i c_j'}{8} t_1^{\lambda-1-\epsilon}$$

so $t_1 \in \mathcal{T}_j(c_j'', \rho_j)$ where $c_j'' = \frac{\rho_i c_j'}{8}$ and $\rho_j = \rho_i/2$ (since $(\rho_i/4)t_1$ needs to be a ρ_j proportion of $t_1/2$).

Finally, we need to show that this probabilistic bound then implies that almost surely there are only finitely many t_1 which are in $\mathcal{T}_i(c_i, \rho_i)$ but not $\mathcal{T}_j(c_j'', \rho_j)$. This case only happens when $\sum_{t=t_1/2}^{(1-\rho_i/4)t_1} \psi_{j,t} \leq \frac{\rho_i c_j'}{8} t_1^{\lambda-\epsilon}$, and the probability of this (by (31)) decreases faster than any inverse polynomial of t_1, and hence has a finite sum over all $t_1 \in \mathcal{T}_i$ (notably, $\sum_{t_1=1}^{\infty} e^{-\frac{\rho_i}{32} c_j' t_1^{\lambda-\epsilon}} < \infty$, so summing only over $t_1 \in \mathcal{T}_i$ must also be finite). Thus, by the Borel-Cantelli Lemma, almost surely it happens only finitely many times, and we are done. ☐

Proposition 4. *For any agent i and constants $c_i, \rho_i > 0$, let $\mathcal{T}_i := \mathcal{T}_i(c_i, \rho_i)$. Then there is a set of constants $c_j, \rho_j > 0$ for all $j \neq i$ such that, almost surely, there is some $t^* > 0$ such that*

$$t_1 > t^* \text{ and } t_1 \in \mathcal{T}_i \implies t_1 \in \mathcal{T}_j \text{ for all } j$$

where $\mathcal{T}_j := \mathcal{T}_j(c_j, \rho_j)$.

Proof. This follows from Lemma 12 by induction over agents sorted by distance to agent i. Let $\mathsf{dist}(i, j)$ denote the distance of vertex j from i, and $N_i(k) := \{j : \mathsf{dist}(i, j) \leq k\}$. We show that if the condition holds for all $j \in N_i(k)$, it holds for all $j \in N_i(k+1)$ as well, and therefore it holds for all $j \in N_i(n)$ (i.e. for all j, since no vertex can be more than n distance from i).

Base case: $k = 0$. Then $N_i(0) = \{i\}$, and the condition is trivially true. Inductive step: we know that there is a $t^*_{(k)}$ such that for all $t_1 > t^*_{(k)}$ and all $j \in N_i(k)$,

$$t_1 \in \mathcal{T}_i \implies t_1 \in \mathcal{T}_j$$

(where $\mathcal{T}_j := \mathcal{T}_j(c_j, \rho_j)$ for $c_j, \rho_j > 0$). Now consider some $j' \in N_i(k+1)$; by definition it has some neighbor $j \in N_i(k)$. Thus by Lemma 12, there is some $t^*(j')$ and constants $c_{j'}, \rho_{j'} > 0$ such that for all $t_1 > t^*(j')$,

$$t_1 \in \mathcal{T}_j \implies t_1 \in \mathcal{T}_{j'}$$

But we know that for all $t_1 > t^*_{(k)}$, $t_1 \in \mathcal{T}_i \implies t_1 \in \mathcal{T}_j$ and therefore for all $t_1 > \max(t^*_{(k)}, t^*(j'))$ we have

$$t_1 \in \mathcal{T}_i \implies t_1 \in \mathcal{T}_{j'}.$$

We can then choose $t^*_{(k+1)} = \max(\max_{j'}(t^*(j')), t^*_{(k)})$ to complete the induction step.

Finally, we take the $c_j, \rho_j > 0$ generated for all j, and, letting $k_{\max} := \max_j \mathsf{dist}(i, j)$, set $t^* = t^*_{(k_{\max})}$ to complete the result. ☐

It is important that the graph is finite, since having only a finite number of induction steps means that the final value of $t^* = t^*_{(k_{\max})}$ will be finite.

We also need that t_1 being a good time for agent i means that $\beta_i(t_1)$ also obeys a constant factor lower bound of order $t_1^{\lambda-1-\epsilon}$ (for all sufficiently large t_1):

Lemma 13. *For any $t_1 \geq 2/\rho_i$, if $t_1 \in \mathcal{T}_i(c_i, \rho_i)$, then*

$$\beta_i(t_1) \geq (c_i/2) t_1^{\lambda-1-\epsilon}.$$

Proof. This follows since $t_1 \in \mathcal{T}_i(c_i, \rho_i)$ and $t_1 \geq 2/\rho_i$ means that there is at least one $t \in [t_1/2, t_1]$ such that

$$\beta_i(t) \geq c_i t^{\lambda-1-\epsilon} \geq c_i t_1^{\lambda-1-\epsilon}.$$

But this means that

$$\begin{aligned}\beta_i(t_1) &= \frac{\sum_{\tau=1}^{t_1} \psi_{i,\tau}}{t_1} \geq \frac{1}{2}\frac{\sum_{\tau=1}^{t_1} \psi_{i,\tau}}{t_1/2} \\ &\geq \frac{1}{2}\frac{\sum_{\tau=1}^{t} \psi_{i,\tau}}{t} = \frac{1}{2}\beta_i(t) \geq (c_i/2) t_1^{\lambda-1-\epsilon}.\end{aligned}$$

□

This yields the result that, almost surely, $\beta_i(t) = \tilde{O}(t^{\lambda-1})$:

Proposition 5. *For any $i \in [n]$ and any $\epsilon > 0$, almost surely*

$$\lim_{t\to\infty} \frac{\beta_i(t)}{t^{\lambda-1+\epsilon}} = 0 \tag{32}$$

$$\lim_{t\to\infty} \frac{\beta_i(t)}{t^{\lambda-1-\epsilon}} = \infty$$

Proof. First, we can show (32) as a corollary of Theorem 5. Since $V(\boldsymbol{\beta}(t)) = \boldsymbol{v}^\top \boldsymbol{\beta}(t) = \sum_{i=1}^n v_i \beta_i(t)$ and each $v_i > 0$ is constant, for each i there is a c_i so that $\beta_i(t) \leq c_i V(\boldsymbol{\beta}(t))$, so

$$\lim_{t\to\infty} \frac{\beta_i(t)}{t^{\lambda-1+\epsilon}} \leq \lim_{t\to\infty} \frac{c_i V(\boldsymbol{\beta}(t))}{t^{\lambda-1+\epsilon}} = 0.$$

Next, Lemma 11 (with $\epsilon/2$) shows that there is some $c > 0$ and time t_0 such that for all $t > t_0$,

$$t \in \mathcal{T}_i(c, 1/n) \text{ for some } i$$

Proposition 4 then shows that for each i, there is some collection of $c_j^{(i)}, \rho_j^{(i)} > 0$ such that almost surely there is a time $t^{(i)}$ such that when $t > t^{(i)}$,

$$t \in \mathcal{T}_i(c, 1/n) \implies t \in \mathcal{T}_j(c_j^{(i)}, \rho_j^{(i)})$$

which, by Lemma 13, implies that (when t is sufficiently large) for any j

$$t \in \mathcal{T}_i(c, 1/n) \implies \beta_j(t) \geq (c_j^{(i)}/2) t^{\lambda-1-\epsilon/2}$$

Finally, let $c_j = \min_i c_j^{(i)} > 0$. Then, for sufficiently large t,

$$\exists i \text{ s.t. } t \in \mathcal{T}_i(c, 1/n) \implies \beta_j(t) \geq (c_j/2) t^{\lambda-1-\epsilon/2}$$

But by Lemma 11 we know that almost surely there is some t_0 such that for any $t > 0$, there is some i such that $t \in \mathcal{T}_i(c, 1/n)$, which then implies that there are $c_1, \ldots, c_n$ such that for any sufficiently large t, $\beta_j(t) \geq (c_j/2) t^{\lambda-1-\epsilon/2}$ and we are done. □

D. Computing Consensus Convergence Rate

Finally, we can use the consensus rate to bound the convergence rate of the inherent belief estimator. If for each i, if $\beta_i(t) \geq c t^{\lambda-1-\epsilon}$, then $\mu_i(t) \geq c t^{\lambda-1-\epsilon}$. Using (18), we have that

$$x(t) \geq (\gamma_i - 1)\mu_i(t) \geq (\gamma_i - 1) c t^{\lambda-1-\epsilon}.$$

Then

$$\begin{aligned} X(t) &\geq (\gamma_i - 1) c \sum_{\tau=t_0}^{t} \tau^{\lambda-1-\epsilon} \\ &\approx (\gamma_i - 1) c \int_{t_0}^{t} \tau^{\lambda-1-\epsilon} d\tau \\ &= (\gamma_i - 1) c \frac{1}{\lambda-\epsilon}\left(t^{\lambda-\epsilon} - t_0^{\lambda-\epsilon}\right) \approx c' t^{\lambda-\epsilon}.\end{aligned}$$

Proposition 3 yields that $\mathbb{P}\left[\exists t \geq t^* : \widehat{\phi}_i(t) \neq \phi_i\right] \leq \delta$ if

$$t^* \geq \left(\frac{1}{c'}\frac{1}{\xi_i}\log\frac{1}{\delta(e^{\xi_i}-1)}\right)^{1/(\lambda-\epsilon)} \tag{33}$$

(assuming that t^* is such that $c'(t^*)^{\lambda-\epsilon} > 2$).

Compared to (22), we see that instead of a rate which is $1/\delta$ to some power, we get $\tilde{\Theta}(\log(1/\delta)^{1/\lambda})$ (since (33) holds for all $\epsilon > 0$), which is a big improvement. This, along with (23), yields the following theorem, suggesting that estimating inherent beliefs in consensus is more difficult:

Theorem 6. *For the inherent belief estimator $\widehat{\phi}_i(t)$ given in Definition 4, let t^* be defined as the time of convergence:*

$$t^* := \max(t : \widehat{\phi}_i(t) \neq \phi_i)$$

i.e. the first time such that $\widehat{\phi}_i(t) = \phi_i$ for all $t > t^*$. Then:

$$t^* = \begin{cases} O(\log(1/\delta)) & \text{if no consensus} \\ O(\log(1/\delta)^{\frac{1}{\lambda}+\epsilon}) \text{ for any } \epsilon > 0 & \text{if consensus}\end{cases}$$

with probability $\geq 1-\delta$, where λ is the largest eigenvalue of $\mathbf{\Gamma}\boldsymbol{W}$ if the consensus is to $\mathbf{0}$, and of $\mathbf{\Gamma}^{-1}\boldsymbol{W}$ if to $\mathbf{1}$.

Proof. This follows directly from (23) (for the non-consensus case) and from (33) (for the consensus case). Note that if something holds for an exponent of $\frac{1}{\lambda-\epsilon}$ for all $\epsilon > 0$, this is equivalent to holding for an exponent of $\frac{1}{\lambda}+\epsilon$ for all $\epsilon > 0$, so we can make the substitution. □

VII. Conclusion

In this work, we study the Interacting Pólya Urn model of opinion dynamics model under social pressure. We expanded upon [1] by showing there exists an estimator for bias parameters and inherent beliefs. Specifically, we showed that the history of any agent and their neighbors' declarations is sufficient in the limit to determine an agent's inherent belief and bias parameter for any network structure, using estimators based on maximum likelihoods. We also analyzed the rate at which the inherent belief estimator converges.

A. Limitations and Future Work

An important open question is how accurately the Interacting Pólya Urn models real social pressure in various situations, in particular how well the proposed interaction between inherent belief (or bias) with social pressure corresponds to reality.

Also, although the model can accommodate general weighted graphs and biases, there is still a significant simplification due to the assumption of a fixed social network (where agents do not change how much they are influenced by others) and synchronous updates. The estimators presented also assume full knowledge of the network and of agents' expressed opinions, which may not be the case when studying real social networks. Furthermore, while this work shows rigorous upper bounds on the (probabilistic) convergence time of the bias parameter and inherent belief estimators, it remains open whether this bound represents the actual convergence time or whether it may converge faster. Finally, the Interacting Pólya Urn model assumes that agents do not change their true belief. A more realistic model may assume that an agent might change her inherent belief (or her bias parameter). New analysis would be required for this, since a constant bias parameter is necessary in this work.

These limitations provide interesting directions for future work. From a social science and psychology perspective, it would be interesting to compare the Interacting Pólya Urn model to the behavior of existing social networks; this may also reveal whether changes to the model may make it a more accurate reflection of real behavior (such as time-discounting, so that opinions expressed further in the past affect the social pressure less than recently-expressed opinions). Furthermore, it remains open if and how the methods presented here can be generalized to variations of the model, for instance if the agents alter the social network over time in response to their social environment. Extending the estimators (and the analysis) to cases of partial information would also increase their practical relevance. Showing lower bounds on the convergence time, and in particular closing the gap to the shown upper bounds, also remains open.

Another crucial area for future work is to consider how to efficiently *intervene* in a social network in order influence declared opinions (for a different interpretation of the model where declared opinions are actions and inherent beliefs are only biases); this stems from one of the key motivations behind the study of opinion dynamics, which is to help guide marketing and persuasion campaigns.

Finally, the model can be extended to $k > 2$ different opinions, with each agent declaring $\psi(t) \in [k]$ at each time, and raises the question of how this would affect the problem of estimating inherent beliefs and biases.

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