Evolution of Opinions under Social Pressure on Random Graphs

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***Abstract*— Opinion dynamics models study how the opinions of individuals evolve in social settings. An important aspect of this often is *social pressure*, in which an individual feels pressure to conform her expressed opinions to the opinions of those around her, even against her true beliefs. This work studies an interacting Pólya urn model for opinion dynamics under social pressure, originally proposed in [1]. In this paper, we consider the behavior of this model on random graphs. Previous work has shown conditions for when the agents on the network approach *consensus* [2], in which all the agents asymptotically express the same opinion over time, even if this opinion is contrary to some of their true beliefs; however these conditions are not interpreted as explicit graph properties or characteristics. In this work, we bridge this gap by examining what kinds of basic network properties determine whether the network approaches consensus. We show that when the agents' network structure is a random graph, *homophily*, the tendency for agents to be connected to those more similar to themselves, diminishes the likelihood of consensus to occur. This result gives insight on how network characteristics affect the possibility of consensus.**

I. INTRODUCTION

Studying opinion dynamics is important for a variety of applications, such as political campaigns, marketing or curbing vaccine hesitancy [3]. While there are many existing models for opinion dynamics, in this work, we study an interacting Pólya urn model of opinion dynamics, which originated in the work of [1] and was subsequently studied by [2]. Key to this model, is that agents have an unchanging *inherent* belief; at each time step they all publicly announce a *declared* opinion which may or may not match their inherent belief. Essentially, the model captures the situation where agents are untruthful due to pressures to conform to their neighbors' opinions. One consequence of this dynamic is the possibility of *consensus*, in which social pressure causes the entire network to converge to (declaring) the majority opinion, even if not all members actually believe in it. Whether the network approaches consensus is a key property of its behavior and affects many other aspects, such as the ease of estimating the agents' true beliefs from their behavior [1].

The interacting Pólya urn model is similar to the Friedkin-Johnsen (another opinion dynamics model discussed in more detail in Section I-B) in that both models update the declared opinion of each agent using both the (declared) opinion of her neighbors as well as a private fixed belief parameter. However, in the Friedkin-Johnsen model, the declared opinions are deterministic and expressed with arbitrary precision as real values, while in the interacting Pólya urn model the agents declare support for one of two basic positions (labeled '0' and '1') randomly based on their private beliefs and the influence of their neighbors.

Previously, in [2], the authors determined the conditions under which the network approaches consensus, which is defined in this model as the proportion of declared opinions equal to '1' approaching either 0 or 1. The conditions are based on the structure of the network and the bias parameters of the agents, but are stated as only algebraic expressions. In this work, we analyze what relations these conditions have with different graph structures and bias parameters, particularly focusing on random graphs.

A. Model Details and Notation

We use the same interacting Pólya urn model as defined in [2]. Readers should refer to [1] and [2] for a more detailed description and justification of the model. Here we summarize the key points and the relevant aspects to our results.

Let (undirected) graph $G = (V, E)$ be a network of n agents (corresponding to the vertices) labeled $i = 1, 2, \ldots, n$. The graph G can have self-loops. For each edge $(i, j) \in E$, there is a weight $a_{i,j} \geq 0$, where by convention we let $a_{i,j} = 0$ if $(i, j) \notin E$. We denote the matrix of these weights as $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, i.e. the weighted adjacency matrix of G; since G is undirected, $\boldsymbol{A}$ is symmetric. We denote the weighted degree of vertex i as $\mathsf{deg}(i) = \sum_j a_{i,j}$, the vector of weighted degrees of all agents as

$$\boldsymbol{d} \triangleq [\mathsf{deg}(1), \mathsf{deg}(2), \ldots, \mathsf{deg}(n)] \tag{1}$$

and its diagonalization as $\boldsymbol{D} = \mathsf{diag}(\boldsymbol{d})$, i.e. the diagonal matrix of the degrees. Let the *normalized adjacency matrix* be $\boldsymbol{W} = \boldsymbol{D}^{-1}\boldsymbol{A}$. The matrix $\boldsymbol{W}$ can be interpreted as the transition matrix for a random walk on G, where the probability of choosing an edge at a given step is proportional to its weight. We assume that $\boldsymbol{W}$ is irreducible (G is connected).

Each agent i has an *inherent belief* $\phi_i \in \{0, 1\}$, which does not change. At each time step t, each agent i (simultaneously) announces a *declared opinion* $\psi_{i,t} \in \{0, 1\}$. Each agent has a bias parameter $\gamma_i \in (0, \infty)$ where $\gamma_i \neq 1$, representing the degree to which the agent favors opinion '1' over opinion '0'. If $\phi_i = 1$, then $\gamma_i > 1$, and if $\phi_i = 0$, then $\gamma_i < 1$.

The declarations $\psi_{i,t}$ are based on a probabilistic rule which we define by the previously observed $\psi_{i,\tau}$ for $\tau < t$. Let $m_i^0, m_i^1 > 0$ represent the initial settings of the model. (Initial settings are used in place of declared opinions at time 1. Some requirements for the initial settings are given shortly.) Define

$$\mu_i^0(t) \triangleq \frac{m_i^0 + \sum_{\tau=2}^{t}\sum_{j=1}^{n} a_{i,j}\mathbb{I}[\psi_{i,\tau}=0]}{m_i^0 + m_i^1 + (t-1)\,\mathsf{deg}(i)} \tag{2}$$

$$\mu_i^1(t) \triangleq \frac{m_i^1 + \sum_{\tau=2}^{t}\sum_{j=1}^{n} a_{i,j}\mathbb{I}[\psi_{i,\tau}=1]}{m_i^0 + m_i^1 + (t-1)\,\mathsf{deg}(i)}\,. \tag{3}$$

The parameter $\mu_i^1(t)$ is essentially the sufficient statistic that summarizes the proportion of declared opinions in the neighborhood of given agent i up to time t. Since $\mu_i^0(t) = 1-\mu_i^1(t)$, we simplify the notation to $\mu_i(t) \triangleq \mu_i^1(t)$.

Define the function (note that μ, γ are scalars)

$$f(\mu,\gamma) \triangleq \frac{\gamma\mu}{1+(\gamma-1)\mu} = \frac{1}{1+\frac{1}{\gamma}\left(\frac{1}{\mu}-1\right)}\,. \tag{4}$$

The probabilistic rule for declared opinions $\psi_{i,t+1}$ can be written as

$$\psi_{i,t+1} \triangleq \begin{cases} 1 & \text{with probability } f(\mu_i(t),\gamma_i) \\ 0 & \text{with probability } 1-f(\mu_i(t),\gamma_i) \end{cases}\,. \tag{5}$$

Note that the bias parameter γ_i is always defined as agent i's bias towards opinion '1'. However, the model is symmetric in the following way: a γ bias towards '1' is equivalent to a $1/\gamma$ bias towards '0', which is captured by the equation $f(\mu_i^1(t),\gamma) = 1 - f(\mu_i^0(t), 1/\gamma)$.

Define the diagonal matrix with $\boldsymbol{\gamma}$ along the diagonal as

$$\boldsymbol{\Gamma} = \mathsf{diag}(\boldsymbol{\gamma})\,. \tag{6}$$

We also define a sufficient statistic that summarizes agent i's declarations. Let $b_i^0, b_i^1 > 0$ (the initialization) be such that $b_i^0 + b_i^1 = 1$ for each i. For $t \in \mathbb{Z}_+$, let

$$\beta_i^0(t) = \frac{b_i^0}{t} + \frac{1}{t}\sum_{\tau=2}^{t}(1-\psi_{i,\tau}) \tag{7}$$

$$\beta_i^1(t) = \frac{b_i^1}{t} + \frac{1}{t}\sum_{\tau=2}^{t}\psi_{i,\tau}\,. \tag{8}$$

These are counts and proportions of declarations of each opinion (or "time-averaged declarations") for each agent (plus initial conditions). We similarly use $\beta_i(t) \triangleq \beta_i^1(t)$. So long as

$$m_i^0 = \sum_{j=1}^{n} a_{i,j} b_j^0 \text{ and } m_i^1 = \sum_{j=1}^{n} a_{i,j} b_j^1\,, \tag{9}$$

it follows from the definition that

$$\mu_i(t) = \frac{1}{\mathsf{deg}(i)}\sum_{j=1}^{n} a_{i,j}\beta_j(t)\,. \tag{10}$$

We define the vectors of all agent's proportions at time t as

$$\boldsymbol{\mu}(t) \triangleq [\mu_1(t), \ldots \mu_n(t)]^\top \tag{11}$$

$$\boldsymbol{\beta}(t) \triangleq [\beta_1(t), \ldots \beta_n(t)]^\top\,. \tag{12}$$

An important term for this work is *consensus*, which needs to be defined appropriately for our stochastic system.

Definition 1: Consensus is approached if

$$\boldsymbol{\beta}(t) \to \mathbf{1} \text{ or } \boldsymbol{\beta}(t) \to \mathbf{0} \quad \text{as} \quad t \to \infty\,. \tag{13}$$

Since $\beta_i(t)$ represents the ratio of '0's agent i has declared, consensus occurs when this ratio goes to 0 or 1.

For (real and symmetric) matrix $\boldsymbol{M}$, let $\lambda_i(\boldsymbol{M})$ be the ith largest eigenvalue of $\boldsymbol{M}$. We will also apply this to real matrices which are not symmetric, but which are similar to a symmetric matrix. Let $\mathbf{1}$ be the all 1's vector and $\mathbf{0}$ be the all 0's vector.

B. Previous Literature

We refer the reader to [1] and [2] for a more detailed review of the literature.

A classic opinion dynamics model is the DeGroot model [4]. Agents are connected on a network and each agent's opinion is represented by a real number. At each time step, every agent averages their neighbors' opinions according to the edge-weights on the network. It was shown that in the DeGroot model, all agents eventually converge to having the same opinion. However, it is not realistic that all agents eventually agree. Many models were developed to understand disagreement among agents. A notable model similar to the DeGroot model is the Friedkin-Johnsen model [5] where agents not only average their neighbors' opinions to update their opinion, but also include their own initial opinion. The interacting Pólya urn model we study has a similar aspect, as the inherent beliefs have a similar role to the initial opinions of the Friedkin-Johnsen model, acting as a constant which affects each update step. Other models similar to the interacting Pólya urn model include those in [6] and [7]. Authors in [6] use a model similar to Friedkin-Johnsen model but where agents have an internal opinion that evolves differently than their external opinion.

In [7], the authors use a model to explain how agents can be pressured to conform to opinions they do not believe in. Though, like the interacting Pólya urn model, their model considers only binary opinions, it has some additional steps. Agents can be either believers or disbelievers of a norm. Due to social pressures, believers can become true believers or false disbelievers, and disbelievers can become true disbelievers or false believers. The model also dictates whether agents want to enforce the norm, an action which pressures their neighbors to comply. Whether an agent chooses to comply or enforce the norm is governed by a threshold function on the agents' observations from their neighbors. The results of [7] were that a small number of true believers can cause a cascade in the population, resulting in an acceptance of a unpopular norm, which occurs if the true believers are clustered together enough. As part of our work, we analyze if an analogous situation occurs in our more streamlined model.

The interacting Pólya urn model for stochastic opinion dynamics was introduced in [1], where authors studied the dynamics when the network is the complete graph and examined their asymptotic behavior. In particular, they considered when it is possible to deduce the inherent beliefs of all agents given only access to the declared opinions and bias parameters. A key result shown is that when consensus occurs, an aggregate estimator is incapable of inferring the inherent beliefs of the agents.

In [2], asymptotic behavior of the dynamics described in Section I-A was studied for general graphs, and it was shown that in arbitrary networks $\boldsymbol{\beta}(t)$ always converges to an *equilibrium point*. Also, the following theorem shown in [2] determines under what conditions (and to which of the two equilibrium points) consensus is achieved:

Define

$$\boldsymbol{J_1} = \boldsymbol{\Gamma}^{-1}\boldsymbol{W} \tag{14}$$

$$\boldsymbol{J_0} = \boldsymbol{\Gamma}\boldsymbol{W}\,. \tag{15}$$

Theorem 1 ([2] Theorem 3): Let $\boldsymbol{x}$ be a boundary equilibrium point (either $\mathbf{0}$ or $\mathbf{1}$). If $\lambda_{\max}(\boldsymbol{J_x}) > 1$, then

$$\mathbb{P}[\boldsymbol{\beta}(t) \to \boldsymbol{x}] = 0. \tag{16}$$

Conversely, if $\lambda_{\max}(\boldsymbol{J_x}) \leq 1$, then

$$\mathbb{P}[\boldsymbol{\beta}(t) \to \boldsymbol{x}] = 1. \tag{17}$$

While this theorem gives a very precise mathematical condition for consensus, it gives little understanding of what kinds of features on networks lead to consensus.

C. Contributions

Using Theorem 1, which characterizes conditions for consensus for the interacting Pólya urn model, this work looks at what specific properties of randomly generated graphs determine whether consensus is approached. Our contributions are:

1) For opinion dynamics on Erdős-Renyi random networks, we show the key to determining whether consensus to $\mathbf{1}$ is approached is to compute a quantity that sums the inverse of all the bias parameters of the agents. If this quantity is less than 1, we can then show that Erdős-Renyi random networks have high probability of approaching consensus.
2) We look at opinion dynamics on a stochastic block model with two communities, one with inherent belief '1' and the other with inherent belief '0'. Similar to the Erdős-Renyi random networks, we determine when agents approach consensus with high probability. The condition to be determined depends on the eigenvalues of a 2×2 matrix. We then restrict the problem to looking at a special case of when all agents have the same expected degree and determine how parameters like the bias parameter, number of agents in each community, and proportion of in-community edges affect whether consensus is approached.

Finally, it is worth mentioning that similar results for consensus to $\mathbf{0}$ can be replicated by simply replacing bias parameter γ_i with $1/\gamma_i$.

II. Eigenvalues of the Laplacian of Random Graphs

In this section, we give a result connecting the eigenvalues of random graph Laplacians to the eigenvalues of $\boldsymbol{\Gamma}^{-1}\boldsymbol{W}$. This will be important for our results on Erdős-Renyi and stochastic block model random graphs. We start with some definitions.

Suppose the adjacency matrix $\boldsymbol{A}$ is generated according to a random graph model (with independent edge probabilities). Let $\bar{\boldsymbol{A}}$ be the *expectation* of $\boldsymbol{A}$. In general, $\bar{\boldsymbol{A}}$ has entries $p_{i,j}$ where $p_{i,j}$ is the probability of an edge occurring between i and j. Let $\bar{\boldsymbol{D}}$ be the diagonal matrix of expected degrees of $\bar{\boldsymbol{A}}$. Specifically, if the entries of $\bar{\boldsymbol{A}}$ are given by $p_{i,j}$ then

$$\bar{\boldsymbol{D}} = \mathsf{diag}\left(\left[\sum_{j=1}^{n} p_{1,j}, \ldots, \sum_{j=1}^{n} p_{n,j}\right]\right). \tag{18}$$

We let $\bar{\boldsymbol{W}} = \bar{\boldsymbol{D}}^{-1}\bar{\boldsymbol{A}}$.

Define $\bar{\boldsymbol{L}} = \boldsymbol{I} - \bar{\boldsymbol{D}}^{-1/2}\bar{\boldsymbol{A}}\bar{\boldsymbol{D}}^{-1/2}$ and $\boldsymbol{L} = \boldsymbol{I} - \boldsymbol{D}^{-1/2}\boldsymbol{A}\boldsymbol{D}^{-1/2}$. These are the expected Laplacian and Laplacian for random graph with adjacency matrix $\boldsymbol{A}$. Let $\|\cdot\|$ be the spectral norm.

Next, we cite a theorem which will be pivotal to our results. This result (combined with Weyl's inequality) will show that the eigenvalues of a random generated edge-independent graph approach the eigenvalues of the expectation of the random graph.

Theorem 2 ([8] Theorem 2): Let G be a random graph where an edge between i and j occurs independently of other edges. Let δ be minimum expected degree of G. For fixed $\epsilon > 0$, there exists a constant $k = k(\epsilon)$ $(k \leq 3)$ such that if $\delta > k\log n$, then with probability at least $1-\epsilon$

$$\|\boldsymbol{L} - \bar{\boldsymbol{L}}\| \leq 3\sqrt{\frac{3\log(4n/\epsilon)}{\delta}}\,. \tag{19}$$

Note that [8] specifically states Theorem 2 for $|\lambda_i(\boldsymbol{L}) - \lambda_i(\bar{\boldsymbol{L}})|$ instead of $\|\boldsymbol{L} - \bar{\boldsymbol{L}}\|$, but uses the inequality

$$|\lambda_i(\boldsymbol{L}) - \lambda_i(\bar{\boldsymbol{L}})| \leq \|\boldsymbol{L} - \bar{\boldsymbol{L}}\| \leq 3\sqrt{\frac{3\log(4n/\epsilon)}{\delta}} \tag{20}$$

to prove their result. (The first inequality is given by Weyl's inequality.) We specifically use the intermediate result of their proofs in our statement of Theorem 2.

There are other similar results which bound the same quantity as Theorem 2 in [8], such as [9]. The result in [10] gives a tighter bound on $\|\boldsymbol{L}-\bar{\boldsymbol{L}}\|$, but it is stated as an almost surely result instead of one which bounds the probability as $1-\epsilon$.

Let $\gamma_{\min} = \min_i \gamma_i$, i.e, the smallest bias parameter of any agent in the network. The following will be the main result we use to analyze the likelihood of reaching consensus for opinion dynamics on random networks.

Proposition 1: Fix the bias parameters of all agents. For a random edge-independent graph G, if the minimum degree δ satisfies $\delta \geq 3\log n$, then with probability at least $1-\epsilon$, we have

$$|\lambda_1(\boldsymbol{\Gamma}^{-1}\boldsymbol{W}) - \lambda_1(\boldsymbol{\Gamma}^{-1}\bar{\boldsymbol{W}})| \leq \frac{1}{\gamma_{\min}} 3\sqrt{\frac{3\log(4n/\epsilon)}{\delta}} \,. \quad (21)$$

Proof:

Substituting in the definitions of $\boldsymbol{L}$ and $\bar{\boldsymbol{L}}$ into Theorem 2

$$\left\| \boldsymbol{D}^{-1/2}\boldsymbol{A}\boldsymbol{D}^{-1/2} - \bar{\boldsymbol{D}}^{-1/2}\bar{\boldsymbol{A}}\bar{\boldsymbol{D}}^{-1/2} \right\| \leq 3\sqrt{\frac{3\log(4n/\epsilon)}{\delta}} \,. \quad (22)$$

For notation, let

$$\boldsymbol{S} = \boldsymbol{D}^{-1/2}\boldsymbol{A}\boldsymbol{D}^{-1/2} \quad (23)$$

$$\bar{\boldsymbol{S}} = \bar{\boldsymbol{D}}^{-1/2}\bar{\boldsymbol{A}}\bar{\boldsymbol{D}}^{-1/2} \,. \quad (24)$$

Then we have

$$\left\| \boldsymbol{\Gamma}^{-1/2}\boldsymbol{S}\boldsymbol{\Gamma}^{-1/2} - \boldsymbol{\Gamma}^{-1/2}\bar{\boldsymbol{S}}\boldsymbol{\Gamma}^{-1/2} \right\| \quad (25)$$

$$= \left\| \boldsymbol{\Gamma}^{-1/2}(\boldsymbol{S} - \bar{\boldsymbol{S}})\boldsymbol{\Gamma}^{-1/2} \right\| \quad (26)$$

$$\leq \|\boldsymbol{\Gamma}^{-1/2}\| \cdot \|\boldsymbol{S} - \bar{\boldsymbol{S}}\| \cdot \|\boldsymbol{\Gamma}^{-1/2}\| \quad (27)$$

$$\leq \frac{1}{\sqrt{\gamma_{\min}}} 3\sqrt{\frac{3\log(4n/\epsilon)}{\delta}} \frac{1}{\sqrt{\gamma_{\min}}} \quad (28)$$

where in the last inequality we used (22) and that

$$\|\boldsymbol{\Gamma}^{-1/2}\| = \frac{1}{\sqrt{\gamma_{\min}}} \,. \quad (29)$$

We get (29) since $\boldsymbol{\Gamma}^{-1/2}$ is diagonal matrix of positive values, so the spectral norm is the largest value.

The matrices $\boldsymbol{\Gamma}^{-1}\boldsymbol{W}$ and $\boldsymbol{\Gamma}^{-1}\bar{\boldsymbol{W}}$ are similar to $\boldsymbol{\Gamma}^{-1/2}\boldsymbol{S}\boldsymbol{\Gamma}^{-1/2}$ and $\boldsymbol{\Gamma}^{-1/2}\bar{\boldsymbol{S}}\boldsymbol{\Gamma}^{-1/2}$ respectively, which means they have the same eigenvalues. Using this and Weyl's inequality gives

$$|\lambda_1(\boldsymbol{\Gamma}^{-1}\boldsymbol{W}) - \lambda_1(\boldsymbol{\Gamma}^{-1}\bar{\boldsymbol{W}})| \quad (30)$$

$$= |\lambda_1(\boldsymbol{\Gamma}^{-1/2}\boldsymbol{S}\boldsymbol{\Gamma}^{-1/2}) - \lambda_1(\boldsymbol{\Gamma}^{-1/2}\bar{\boldsymbol{S}}\boldsymbol{\Gamma}^{-1/2})| \quad (31)$$

$$\leq \left\| \boldsymbol{\Gamma}^{-1/2}\boldsymbol{S}\boldsymbol{\Gamma}^{-1/2} - \boldsymbol{\Gamma}^{-1/2}\bar{\boldsymbol{S}}\boldsymbol{\Gamma}^{-1/2} \right\| \quad (32)$$

$$\leq \frac{1}{\gamma_{\min}} 3\sqrt{\frac{3\log(4n/\epsilon)}{\delta}} \,. \quad (33)$$

■

Corollary 1: Fix the bias parameters, n and $\delta > 3\log n$, where δ is the minimum expected degree. Suppose the largest eigenvalue of $\boldsymbol{\Gamma}^{-1}\bar{\boldsymbol{W}}$ is λ where $\lambda < 1$ and $1-\lambda = \Delta$. Then for a randomly generated edge-independent graph,

$$\mathbb{P}[\boldsymbol{\beta}(t) \to \mathbf{1}] \geq 1 - 4n\exp\left(-\frac{\delta\Delta^2\gamma_{min}^2}{27}\right) \,. \quad (34)$$

Proof: As a probability over the randomly generated graph

$$\mathbb{P}[\boldsymbol{\beta}(t) \to \mathbf{1}] = \mathbb{P}[\lambda_1(\boldsymbol{\Gamma}^{-1}\boldsymbol{W}) < 1] \quad (35)$$

$$\geq \mathbb{P}[|\lambda_1(\boldsymbol{\Gamma}^{-1}\boldsymbol{W}) - \lambda_1(\boldsymbol{\Gamma}^{-1}\bar{\boldsymbol{W}})| \leq \Delta] \quad (36)$$

$$\geq 1 - \epsilon \quad (37)$$

if

$$\Delta \geq \frac{1}{\gamma_{\min}} 3\sqrt{\frac{3\log(4n/\epsilon)}{\delta}} \,. \quad (38)$$

which holds if ϵ satisfies

$$\epsilon \geq 4n\exp\left(-\frac{\delta\Delta^2\gamma_{min}^2}{27}\right) \,. \quad (39)$$

Choosing ϵ for which equality holds completes the proof. ■

If in the random graphs we are considering, $\delta = \Omega(n)$, then when $\boldsymbol{\Gamma}^{-1}\bar{\boldsymbol{W}}$ has eigenvalues less than $1-\Delta$ for $\Delta > 0$, the probability that consensus goes to 1 is exponentially fast with n. If we change the bound to $\delta = \omega(\log n)$, then the probability of consensus still goes to 1 under the same conditions, however, the decay may not be exponential.

III. Erdős-Renyi Random Graphs

Let $G_{ER}(n,p)$ be a randomly generated Erdős-Renyi graph on n nodes and (independent) edge probabilities p. (Recall that each node can have an edge with itself.) Let $\boldsymbol{A} \sim G_{ER}(n,p)$ be the adjacency matrix of a random graph from the distribution $G_{ER}(n,p)$. Then $\bar{\boldsymbol{A}}$ is simply the matrix will all entries p and $\bar{\boldsymbol{D}}$ is a diagonal matrix with entries np.

Lemma 1: For $\boldsymbol{A} \sim G(n,p)$ and a matrix of bias parameters $\boldsymbol{\Gamma}$, we have

$$\lambda_1(\boldsymbol{\Gamma}^{-1}\bar{\boldsymbol{W}}) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\gamma_i} \,. \quad (40)$$

Proof: First note that $\boldsymbol{\Gamma}^{-1}\bar{\boldsymbol{W}}$ is diagonalizable since it is similar to a symmetric matrix.

We can write

$$\boldsymbol{\Gamma}^{-1}\bar{\boldsymbol{W}} = \boldsymbol{\Gamma}^{-1}\frac{p}{pn}\mathbf{1}\mathbf{1}^\top = \frac{1}{n}\boldsymbol{\Gamma}^{-1}\mathbf{1}\mathbf{1}^\top \,. \quad (41)$$

Matrix $\boldsymbol{\Gamma}^{-1}\bar{\boldsymbol{W}}$ must have rank 1 as each row is proportional to $\mathbf{1}^\top$. Thus it must only have one nonzero eigenvalue.

Consider $\boldsymbol{x} = \boldsymbol{\Gamma}^{-1}\mathbf{1}$. Then

$$\boldsymbol{\Gamma}^{-1}\bar{\boldsymbol{W}}\boldsymbol{x} = \frac{1}{n}\boldsymbol{\Gamma}^{-1}\mathbf{1}\mathbf{1}^\top\boldsymbol{\Gamma}^{-1}\mathbf{1} \quad (42)$$

$$= \frac{1}{n}\boldsymbol{\Gamma}^{-1}\mathbf{1}\sum_{i=1}^{n}\frac{1}{\gamma_i} \quad (43)$$

$$= \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\gamma_i}\right)\boldsymbol{x} \,. \quad (44)$$

Hence $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{\Gamma}^{-1}\bar{\boldsymbol{W}}$ with an eigenvalue of $\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\gamma_i}$. This is the only nonzero eigenvalue so it must also be the largest eigenvalue, which gives the result. ■

Theorem 3: Suppose each agent has bias parameter $\gamma_i > \gamma_{\min}$. For an Erdős-Renyi randomly generated graph with edge probabilities $p \geq (3\log n)/n$, if

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\gamma_i} < 1 - \Delta \quad (45)$$

then

$$\mathbb{P}_{A\sim G(n,p)}[\boldsymbol{\beta}(t) \to \mathbf{1}] \geq 1 - 4n\exp\left(-\frac{np\Delta^2\gamma_{\min}^2}{27}\right) \,. \quad (46)$$

Proof: We use Corollary 1 combined with Lemma 1. We can let $\delta = np$ since all nodes have the same expected degree. ■

Theorem 3 shows that if all agents are equally likely to share an edge with any other agent, the only quantity that governs whether consensus is approached with high probability is given by looking at the sum of the inverse of the bias parameters. This directly shows how to link the values of the bias parameters to the likelihood of consensus being approached. If p is not constant, say $p = c(\log n)/n$, then the constant c would need to be large enough compared to Δ^2 and $\gamma_{\min}^2$ to ensure the probability of approaching consensus goes to 1.

IV. Stochastic Block Models

In this section, we will study in detail conditions for approaching consensus for the stochastic block model with two communities. The two communities are labeled A and B. Of the n agents, n_A are in community A and the remaining n_B are in community B. The set of indices which correspond to agents in community A and B are V_A and V_B respectively.

All agents in community A has a bias towards opinion 1. We will use the simplifying assumption that the bias parameter of all agents in A is γ_A where $\gamma_A > 1$. Similarly the bias parameter of all agents in B is γ_B where $\gamma_B < 1$.

Let p_A be the probability of an edge occurring between two agents in community A, and p_B be the same probability for agents in community B. (Recall again that we allow edges from an agent to herself.) Let q be the probability of an edge occurring between an agent in A and an agent in B. We denote the random graph model using these parameters as $G_{SB}(n_A, n_B, p_A, p_B, q)$.

The values p_A, p_B, and q are present in the expected adjacency matrix. For example, if $\boldsymbol{A} \sim G_{SB}(3, 2, p_A, p_B, q)$, then if $V_A = \{1, 2, 3\}$ and $V_B = \{4, 5\}$, we have

$$\bar{\boldsymbol{A}} = \begin{bmatrix} p_A & p_A & p_A & q & q \\ p_A & p_A & p_A & q & q \\ p_A & p_A & p_A & q & q \\ q & q & q & p_B & p_B \\ q & q & q & p_B & p_B \end{bmatrix} . \tag{47}$$

The matrix $\bar{\boldsymbol{W}} = \bar{\boldsymbol{D}}^{-1}\bar{\boldsymbol{A}}$ contains four different entries, where

$$\bar{w}_{i,j} = \begin{cases} \frac{p_A}{p_A n_A + q n_B} & \text{if } i \in V_A, j \in V_A \\ \frac{q}{p_A n_A + q n_B} & \text{if } i \in V_A, j \in V_B \\ \frac{q}{q n_A + p_B n_B} & \text{if } i \in V_B, j \in V_A \\ \frac{p_B}{q n_A + p_B n_B} & \text{if } i \in V_B, j \in V_B \end{cases} . \tag{48}$$

A. Conditions for Approaching Consensus

To find conditions on approaching consensus for random graphs $\boldsymbol{A} \sim G_{SB}(n_A, n_B, p_A, p_B, q)$, we analyze the eigenvalues of matrix $\boldsymbol{\Gamma}^{-1}\bar{\boldsymbol{W}}$. For fixed model $G_{SB}(n_A, n_B, p_A, p_B, q)$ and bias parameters γ_A and γ_B for community A agents and B agents respectively, let

$$\boldsymbol{J}_{\mathbf{1}}^{(2)} = \begin{bmatrix} \frac{1}{\gamma_A} & 0 \\ 0 & \frac{1}{\gamma_B} \end{bmatrix} \begin{bmatrix} \frac{p_A n_A}{p_A n_A + q n_B} & \frac{q n_B}{p_A n_A + q n_B} \\ \frac{q n_A}{p_B n_B + q n_A} & \frac{p_B n_B}{p_B n_B + q n_A} \end{bmatrix} . \tag{49}$$

Proposition 2: Matrix $\boldsymbol{\Gamma}^{-1}\bar{\boldsymbol{W}}$ has two nonzero eigenvalues, which are the same as the eigenvalues of the 2×2 matrix $\boldsymbol{J}_{\mathbf{1}}^{(2)}$.

Proof: Since $\boldsymbol{\Gamma}^{-1}\bar{\boldsymbol{W}}$ is similar to a symmetric matrix $\boldsymbol{\Gamma}^{-1/2}\bar{\boldsymbol{D}}^{-1/2}\bar{\boldsymbol{A}}\bar{\boldsymbol{D}}^{-1/2}\boldsymbol{\Gamma}^{-1/2}$, we can conclude that $\boldsymbol{\Gamma}^{-1}\bar{\boldsymbol{W}}$ is diagonalizable and its rank is equivalent to its number of nonzero eigenvalues. Since $\mathsf{rank}(\boldsymbol{\Gamma}^{-1}\bar{\boldsymbol{W}}) = 2$, we can conclude there are only two nonzero eigenvalues.

Matrix $\boldsymbol{J}_{\mathbf{1}}^{(2)}$ has two eigenvalues. Suppose that one eigenvalue-eigenvector pair is λ and $\boldsymbol{v} = [v_1, v_2]^\top$. Then

$$\boldsymbol{J}_{\mathbf{1}}^{(2)}\boldsymbol{v} = \lambda \boldsymbol{v} \tag{50}$$

$$\implies \begin{cases} \frac{1}{\gamma_A} v_1 \frac{p_A n_A}{p_A n_A + q n_B} + \frac{1}{\gamma_A} v_2 \frac{q n_B}{p_A n_A + q n_B} & = \lambda v_1 \\ \frac{1}{\gamma_B} v_1 \frac{q n_A}{p_B n_B + q n_A} + \frac{1}{\gamma_B} v_2 \frac{p_B n_B}{p_B n_B + q n_A} & = \lambda v_2 \end{cases} . \tag{51}$$

Now let $\boldsymbol{x}$ be an n-length vector where

$$x_i = \begin{cases} v_1 & \text{if } i \in V_A \\ v_2 & \text{if } i \in V_B \end{cases} . \tag{52}$$

Then (let $(\boldsymbol{\Gamma}^{-1}\bar{\boldsymbol{W}}\boldsymbol{x})_i$ denote the i entry of vector $\boldsymbol{\Gamma}^{-1}\bar{\boldsymbol{W}}\boldsymbol{x}$)

$$(\boldsymbol{\Gamma}^{-1}\bar{\boldsymbol{W}}\boldsymbol{x})_i = \begin{cases} \frac{1}{\gamma_A} \frac{v_1 p_A n_A + v_2 q n_B}{p_A n_A + q n_B} & \text{if } i \in V_A \\ \frac{1}{\gamma_B} \frac{v_1 q n_A + v_2 p_B n_B}{p_B n_A + q n_A} & \text{if } i \in V_B \end{cases} \tag{53}$$

$$= \begin{cases} \lambda v_1 & \text{if } i \in V_A \\ \lambda v_2 & \text{if } i \in V_B \end{cases} \tag{54}$$

which means that

$$\boldsymbol{\Gamma}^{-1}\bar{\boldsymbol{W}}\boldsymbol{x} = \lambda \boldsymbol{x} \tag{55}$$

and thus λ is an eigenvalue of $\boldsymbol{\Gamma}^{-1}\bar{\boldsymbol{W}}$. ■

Theorem 4: For fixed model $G_{SB}(n_A, n_B, p_A, p_B, q)$ and bias parameters γ_A and γ_B for community A agents and B agents respectively, if the larger eigenvalues of $\boldsymbol{J}_{\mathbf{1}}^{(2)}$ is at least Δ less than 1, then

$$\mathbb{P}_{A \sim G_{SB}(n_A, n_B, p_A, p_B, q)}[\boldsymbol{\beta}(t) \to \mathbf{1}] \tag{56}$$

$$\geq 1 - 4n \exp\left(-\frac{\delta \Delta^2 \gamma_{\min}^2}{27}\right) \tag{57}$$

if $\delta = \min\{p_A n_A + q n_B, p_B n_B + q n_A\} \geq 3 \log n$.

Proof: Using Corollary 1 with Proposition 2 gives the result. The expected degree of agents in community A is given by $p_A n_A + q n_B$ and that of agents in community B is given by $p_B n_B + q n_A$. Taking the minimum of the two gives the value for δ. ■

Whether the largest eigenvalue of a 2×2 is less than 1 can be given by the following condition. Suppose we have matrix

$$\boldsymbol{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} . \tag{58}$$

Then the largest eigenvalue of $\boldsymbol{M}$ is less than 1 if

$$0 < a < 1 \tag{59}$$

$$0 < d < 1 \tag{60}$$

$$\text{and } bc < (1-a)(1-d) . \tag{61}$$

B. Equal Expected Degree

To gain some intuition about what kinds of random two community networks approach consensus with high probability and which do not, we will look at the case when the averaged expected degree of agents in both community A and community B are the same. This implies that

$$p_A n_A + q n_B = q n_A + p_B n_B \triangleq d\,. \tag{62}$$

Proposition 3: In the random two community model where all agents have the same expected degree $d \geq 3\log n$ and $\gamma_B < 1 < \gamma_A$, the largest eigenvalue of $\boldsymbol{J}_{\mathbf{1}}^{(2)}$ is less than 1 if all the following hold:

$$0 < n_B(\gamma_B - 1) + n_A(\gamma_A - 1) \tag{63}$$

$$\frac{p_B n_B}{d} < \gamma_B \tag{64}$$

$$\frac{p_A n_A}{d} < \frac{\gamma_A n_B(\gamma_B - 1) + n_A(\gamma_A - 1)}{n_B(\gamma_B - 1) + n_A(\gamma_A - 1)}\,. \tag{65}$$

Proof: Let d_{AB} be the expected number of edges each agent in A has with agents in B and d_{BA} be the expected number of edges each agent in B has with agents in A. The total number of edges between A and B is expressed as

$$n_B d_{BA} = n_A d_{AB} \implies d_{BA} = \frac{n_A}{n_B} d_{AB}\,. \tag{66}$$

Let $p' = \frac{p_A n_A}{d}$. Then we get that

$$\bar{\boldsymbol{A}}^{(2)} = \begin{bmatrix} \frac{p_A n_A}{d} & \frac{q n_B}{d} \\ \frac{q n_A}{d} & \frac{p_B n_B}{d} \end{bmatrix} \tag{67}$$

$$= \begin{bmatrix} \frac{p_A n_A}{d} & \frac{d - p_A n_A}{d} \\ \frac{n_A}{n_B}\frac{d - p_A n_A}{d} & 1 - \frac{n_A}{n_B}\frac{d - p_A n_A}{d} \end{bmatrix} \tag{68}$$

$$= \begin{bmatrix} p' & 1 - p' \\ \frac{n_A}{n_B}(1 - p') & 1 - \frac{n_A}{n_B}(1 - p') \end{bmatrix} \tag{69}$$

$$\boldsymbol{J}_{\mathbf{1}}^{(2)} = \begin{bmatrix} \frac{1}{\gamma_A} p' & \frac{1}{\gamma_A}(1 - p') \\ \frac{1}{\gamma_B}\frac{n_A}{n_B}(1 - p') & \frac{1}{\gamma_B}\left(1 - \frac{n_A}{n_B}(1 - p')\right) \end{bmatrix}. \tag{70}$$

Applying (60) gives the condition (64) of the result. (Note that applying (59) gives $\frac{p_A n_A}{d} < \gamma_A$ which is always true since $\gamma_A > 1$.) By (61), one of the following needs to hold:

$$\begin{cases} 0 & < n_B(\gamma_B - 1) + n_A(\gamma_A - 1) \\ p' & < \frac{\gamma_A n_B(\gamma_B - 1) + n_A(\gamma_A - 1)}{n_B(\gamma_B - 1) + n_A(\gamma_A - 1)} \end{cases} \tag{71}$$

$$\text{or } \begin{cases} 0 & > n_B(\gamma_B - 1) + n_A(\gamma_A - 1) \\ p' & > \frac{\gamma_A n_B(1 - \gamma_B) + n_A(1 - \gamma_A)}{n_B(1 - \gamma_B) + n_A(1 - \gamma_A)} \end{cases}. \tag{72}$$

However, notice that in the second item, since

$$n_B(1 - \gamma_B) > -n_A(1 - \gamma_A) \tag{73}$$

$$n_B(1 - \gamma_B) > 0 \tag{74}$$

and since $\gamma_A > 1$, this implies

$$\gamma_A n_B(1 - \gamma_B) > n_B(1 - \gamma_B) \tag{75}$$

and thus the right-hand side of the second equation in (72) is always greater than 1. Since $p' < 1$, this condition never occurs. Thus, we need to satisfy (71). ∎

We note that Proposition 3 gives some interesting insights on when consensus can be achieved. First, we see that it is certainly possible to approach a consensus of '1' even if there are fewer agents who have inherent belief '1'. The necessary condition is that (63) needs to hold. Given any setting of n_B and n_A and p_A, we can always increase γ_A so that both (63) and (65) hold. Conceptually, increasing γ_A will always increases the number of declared '1's. Exactly what to increase γ_A is given by (65).

Second, Proposition 3 explains the effects of *homophily*, the tendency for agents to be connected to those more similar to themselves. Both the quantities p_A and p_B represents probabilities of edges within the same community of an agent. We can observe the effect of the homophily in our model by fixing n_A, n_B and d. The Erdős-Renyi (no homophily) corresponds to $p_A = p_B = q = \frac{d}{n_A + n_B}$ (see (62)). Increasing q results in lower values for both p_A and p_B. Proposition 3 indicates that *lower* values of p_A and p_B make it easier for consensus to occur. This brings about the following point: When agents are connected with those more similar to them, it is harder for consensus to be approached. This is different from the observations made in [7]. Their conclusions were that a cascade of complying with a unpopular norm (a situation which parallels approaching consensus) occurs when a small number of true believers of the norm are clustered together (a situation which parallels p_A being larger). As we do not get this in our two community model, this leads to the conclusion that the different aspects of the model in [7], key among them being the threshold function and the enforcement step, do affect the cascades of opinions.

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