Embedding Formulations and Complexity for Unions of Polyhedra

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(Linear) Mixed 0-1 Integer Formulations

- Modeling Finite Alternatives = Unions of Polyhedra

\[ x \in \bigcup_{i=1}^{n} P_i \subseteq \mathbb{R}^d \]
(Linear) Mixed 0-1 Integer Formulations

• Modeling Finite Alternatives = Unions of Polyhedra

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\[ \text{gr} (f) = \bigcup_{i=1}^{n} P_i \]
(Linear) Mixed 0-1 Integer Formulations

- Modeling Finite Alternatives = Unions of Polyhedra

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\[ \text{gr } (f) = \bigcup_{i=1}^{n} P_i \]
(Linear) Mixed 0-1 Integer Formulations

- Modeling Finite Alternatives = Unions of Polyhedra

\[
\begin{align*}
\min & \quad \sum_{j=1}^{m} f_j(x_j, y_j) \\
\text{s.t.} & \quad (x, y) \in X
\end{align*}
\]

\[\text{gr}(f) = \bigcup_{i=1}^{n} P_i\]
Size of Smallest 0-1 Formulation for \( x \in \bigcup_{i=1}^{n} P_i \)

- Standard **ideal (integral) extended** formulation for

\[
P_i = \{ x \in \mathbb{R}^d : A^i x \leq b^i \} \quad \text{(Balas, Jeroslow and Lowe):}
\]

\[
A^i x^i \leq b^i y_i \quad \forall i \in \{1, \ldots, n\}
\]

\[
\sum_{i=1}^{n} x^i = x, \quad x^i \in \mathbb{R}^d \quad \forall i \in \{1, \ldots, n\}
\]

\[
\sum_{i=1}^{n} y_i = 1, \quad y \in \{0, 1\}^n
\]
Size of Smallest 0-1 Formulation for $x \in \bigcup_{i=1}^{n} P_i$

- Standard **ideal (integral) extended** formulation for $\quad P_i = \{ x \in \mathbb{R}^d : A^i x \leq b^i \}$ (Balas, Jeroslow and Lowe):

  $$A^i x^i \leq b^i y_i \quad \forall i \in \{1, \ldots, n\}$$

  $$\sum_{i=1}^{n} x^i = x, \quad x^i \in \mathbb{R}^d \quad \forall i \in \{1, \ldots, n\}$$

  $$\sum_{i=1}^{n} y_i = 1, \quad y \in \{0, 1\}^n$$

- What about non-extended (i.e. no variables copies)?
Size of Smallest 0-1 Formulation for $x \in \bigcup_{i=1}^{n} P_i$

- Standard **ideal (integral) extended** formulation for $P_i = \{ x \in \mathbb{R}^d : A^i x \leq b^i \}$ (Balas, Jeroslow and Lowe):
  
  $A^i x^i \leq b^i y_i \quad \forall i \in \{1, \ldots, n\}$

  $\sum_{i=1}^{n} x^i = x, \quad x^i \in \mathbb{R}^d \quad \forall i \in \{1, \ldots, n\}$

  $\sum_{i=1}^{n} y_i = 1, \quad y \in \{0, 1\}^n$

- What about non-extended (i.e. no variables copies)?
- What about non-ideal? (i.e. some fractional extreme pts.)?
Size of Smallest 0-1 Formulation for $x \in \bigcup_{i=1}^{n} P_i$

- Standard **ideal (integral) extended** formulation for

  $$P_i = \{ x \in \mathbb{R}^d : A^i x \leq b^i \}$$ (Balas, Jeroslow and Lowe):

  $$A^i x^i \leq b^i y_i$$ \quad $\forall i \in \{1, \ldots, n\}$

  $$\sum_{i=1}^{n} x^i = x, \quad x^i \in \mathbb{R}^d \quad \forall i \in \{1, \ldots, n\}$$

  $$\sum_{i=1}^{n} y_i = 1, \quad y \in \{0, 1\}^n$$

- What about non-extended (i.e. no variables copies)？
- What about non-ideal? (i.e. some fractional extreme pts.)？
- What about **precise** lower/upper bounds on size?
Performance for Univariate Functions

- Results from Nemhauser, Ahmed, and V. ’10 using CPLEX 11

- Non-extended and ideal formulations provide a significant computational advantage
Constructing Non-extended Ideal Formulations

• Pure Integer:
Constructing Non-extended Ideal Formulations

- Pure Integer:

\[
Q := \text{conv}\left( \left\{ p^i \right\}_{i=1}^n \right)
\]
Constructing Non-extended Ideal Formulations

- Pure Integer:

\[ Q := \text{conv} \left( \{ p^i \}_{i=1}^n \right) \]

\[ Q \cap \mathbb{Z}^2 \]
Constructing Non-extended Ideal Formulations

- Pure Integer:
  \[ Q := \text{conv} \left( \{ p^i \}_{i=1}^n \right) \]

- Mixed Integer:

\[ Q \cap \mathbb{Z}^2 \]

\[ P_1 \]

\[ P_2 \]

\[ P_3 \]
Constructing Non-extended Ideal Formulations

- **Pure Integer:**

\[ Q := \text{conv} \left( \{ p^i \}_{i=1}^n \right) \]

- **Mixed Integer:**

\[ Q \cap \mathbb{Z}^2 \]

Embedding Formulations
"Simple" Family of Polyhedra

\[(x, z) \in \text{gr}(f) = \bigcup_{i=1}^{3} P_i\]
“Simple” Family of Polyhedra

\[(x, z) \in \text{gr}(f) = \bigcup_{i=1}^{3} P_i\]

\[
\begin{pmatrix}
    x \\
    z
\end{pmatrix} = \sum_{j=1}^{4} \begin{pmatrix}
    d_j \\
    f(d_j)
\end{pmatrix} \lambda d_j
\]

\[
\lambda \in \Delta^4 := \left\{ \lambda \in \mathbb{R}_+^4 : \sum_{i=1}^{4} \lambda_i = 1 \right\}
\]
“Simple” Family of Polyhedra

\[(x, z) \in \text{gr}(f) = \bigcup_{i=1}^{3} P_i\]

\[
\begin{pmatrix} x \\ z \end{pmatrix} = \sum_{j=1}^{4} \begin{pmatrix} d_j \\ f(d_j) \end{pmatrix} \lambda d_j
\]

\[\lambda \in \Delta^4 := \left\{ \lambda \in \mathbb{R}^4_+ : \sum_{i=1}^{4} \lambda_i = 1 \right\}\]
"Simple" Family of Polyhedra

\[(x, z) \in \text{gr} (f) = \bigcup_{i=1}^{3} P_i\]

\[
\begin{pmatrix}
  x \\
  z
\end{pmatrix} = \sum_{j=1}^{4} \left( \frac{d_j}{f(d_j)} \right) \lambda d_j
\]

\[
\lambda \in \Delta^4 := \left\{ \lambda \in \mathbb{R}_+^4 : \sum_{i=1}^{4} \lambda_i = 1 \right\}
\]
“Simple” Family of Polyhedra

\[(x, z) \in \text{gr}(f) = \bigcup_{i=1}^{3} P_i\]

\[
\begin{pmatrix} x \\ z \end{pmatrix} = \sum_{j=1}^{4} \binom{d_j}{f(d_j)} \lambda_{d_j}
\]

\[
\lambda \in \Delta^4 := \left\{ \lambda \in \mathbb{R}^4_+ : \sum_{i=1}^{4} \lambda_i = 1 \right\}
\]
“Simple” Family of Polyhedra

\[(x, z) \in \text{gr}(f) = \bigcup_{i=1}^{3} P_i\]

\[
\begin{pmatrix}
  x \\
  z
\end{pmatrix} = \sum_{j=1}^{4} \left( \frac{d_j}{f(d_j)} \right) \lambda_{d_j}
\]

\[
\lambda \in \Delta^4 := \left\{ \lambda \in \mathbb{R}_+^4 : \sum_{i=1}^{4} \lambda_i = 1 \right\}
\]

\[
\lambda \in \bigcup_{i=1}^{3} P_i \subseteq \Delta^4
\]

\[
P_i := \left\{ \lambda \in \Delta^4 : \lambda_d = 0 \quad \forall d \notin T_i \right\}
\]

\[
T_i := \{d_i, d_{i+1}\} \quad i \in \{1, \ldots, 3\}
\]
“Simple” Family of Polyhedra

\[(x, z) \in \text{gr } (f) = \bigcup_{i=1}^{3} P_i\]

SOS2 Constraints

\[
\begin{pmatrix} x \\ z \end{pmatrix} = \sum_{j=1}^{4} \begin{pmatrix} d_j \\ f(d_j) \end{pmatrix} \lambda d_j
\]

\[\lambda \in \Delta^4 := \left\{ \lambda \in \mathbb{R}_+^4 : \sum_{i=1}^{4} \lambda_i = 1 \right\}\]

\[\lambda \in \bigcup_{i=1}^{3} P_i \subseteq \Delta^4\]

\[P_i := \left\{ \lambda \in \Delta^4 : \lambda_d = 0 \quad \forall d \notin T_i \right\}\]

\[T_i := \{d_i, d_{i+1}\} \quad i \in \{1, \ldots, 3\}\]
“Simple” Family of Polyhedra

\[(x, z) \in \text{gr} (f) = \bigcup_{i=1}^{3} P_i\]

\[
f(d_4)
\]

\[
f(d_3)
\]

\[
f(d_2)
\]

\[
f(d_1)
\]

\[
\begin{align*}
\lambda & \in \bigcup_{i=1}^{3} P_i \subseteq \Delta^4 \\
&P_i := \{\lambda \in \Delta^4 : \lambda_d = 0 \quad \forall d \notin T_i\} \\
&T_i := \{d_i, d_{i+1}\} \quad i \in \{1, \ldots, 3\}
\end{align*}
\]

SOS2 Constraints
Standard **Non-ideal** Formulation for SOS2

\[
\begin{align*}
2(n + 1) & \\
0 & \leq \lambda_1 \leq y_1 \\
0 & \leq \lambda_2 \leq y_1 + y_2 \\
0 & \leq \lambda_3 \leq y_2 + y_3 \\
0 & \leq \lambda_4 \leq y_3 + y_4 \\
0 & \leq \lambda_5 \leq y_4
\end{align*}
\]

\[
\sum_{i=1}^{5} \lambda_i = 1 \quad \text{and} \quad y \in \{0, 1\}^4, \quad \sum_{i=1}^{4} y_i = 1
\]
Standard **Non-ideal** Formulation for SOS2

\[ T_1 \quad T_2 \quad T_3 \quad T_4 \]

\[ 1 \quad 2 \quad 3 \quad 4 \quad 5 = n + 1 \]

\[ 2(n + 1) \]

\[ 0 \leq \lambda_1 \leq y_1 \]
\[ 0 \leq \lambda_2 \leq y_1 + y_2 \]
\[ 0 \leq \lambda_3 \leq y_2 + y_3 \]
\[ 0 \leq \lambda_4 \leq y_3 + y_4 \]
\[ 0 \leq \lambda_5 \leq y_4 \]

\[ \sum_{i=1}^{5} \lambda_i = 1 \]
\[ y \in \{0, 1\}^4, \quad \sum_{i=1}^{4} y_i = 1 \]

**General Inequalities**
Standard **Non-ideal** Formulation for SOS2

\[
T_1 \rightarrow T_2 \rightarrow T_3 \rightarrow T_4
\]

\[
1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 = n + 1
\]

\[
2(n + 1)
\]

\[
0 \leq \lambda_1 \leq y_1
\]

\[
0 \leq \lambda_2 \leq y_1 + y_2
\]

\[
0 \leq \lambda_3 \leq y_2 + y_3
\]

\[
0 \leq \lambda_4 \leq y_3 + y_4
\]

\[
0 \leq \lambda_5 \leq y_4
\]

\[
\sum_{i=1}^{5} \lambda_i = 1
\]

\[
y \in \{0, 1\}^4, \quad \sum_{i=1}^{4} y_i = 1
\]

General Inequalities
Standard **Non-ideal** Formulation for SOS2

\[
\begin{align*}
\sum_{i=1}^{5} \lambda_i &= 1 \\
y &\in \{0, 1\}^4, \quad \sum_{i=1}^{4} y_i = 1
\end{align*}
\]

- Minimum # of *(general)* inequalities?
  - Ideal formulation:
  - Non-ideal formulation:

\[
\begin{align*}
0 &\leq \lambda_1 \leq y_1 \\
0 &\leq \lambda_2 \leq y_1 + y_2 \\
0 &\leq \lambda_3 \leq y_2 + y_3 \\
0 &\leq \lambda_4 \leq y_3 + y_4 \\
0 &\leq \lambda_5 \leq y_4
\end{align*}
\]
Standard **Non-ideal** Formulation for SOS2

\[ T_1 \rightarrow T_2 \rightarrow T_3 \rightarrow T_4 \quad 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \quad 5 = n + 1 \]

\[ 2(n + 1) \]

\[ \sum_{i=1}^{5} \lambda_i = 1 \]

\[ y \in \{0, 1\}^4, \quad \sum_{i=1}^{4} y_i = 1 \]

- **Minimum # of (general) inequalities?**
  - **Ideal formulation:**
    \[ 2\lceil \log_2 n \rceil \]
    \[ n + 1 \leq \ldots \leq n + 1 + 2\lceil \log_2 n \rceil \]
  - **Non-ideal formulation:**

  **General Inequalities**
Standard **Non-ideal** Formulation for SOS2

\[ T_1 \quad T_2 \quad T_3 \quad T_4 \]

\[ 1 \quad 2 \quad 3 \quad 4 \quad 5 = n + 1 \]

\[ 2(n + 1) \]

\[ 0 \leq \lambda_1 \leq y_1 \]

\[ 0 \leq \lambda_2 \leq y_1 + y_2 \]

\[ 0 \leq \lambda_3 \leq y_2 + y_3 \]

\[ 0 \leq \lambda_4 \leq y_3 + y_4 \]

\[ 0 \leq \lambda_5 \leq y_4 \]

\[ \sum_{i=1}^{4} y_i = 1 \]

\[ \sum_{i=1}^{5} \lambda_i = 1 \]

- Minimum # of (general) inequalities?
  - Ideal formulation:
    \[ 2 \lfloor \log_2 n \rfloor \]
    \[ n + 1 \leq \ldots \leq n + 1 + 2 \lfloor \log_2 n \rfloor \]
  - Non-ideal formulation:
    \[ 2 \leq \ldots \leq 4 \]
    \[ 2 \leq \ldots \leq 5 + 2n \]
What is a Formulation?

\[
\begin{align*}
T_1 & \quad T_2 \quad T_3 \quad T_4 \\
1 & \quad 2 \quad 3 \quad 4 \quad 5
\end{align*}
\]

\[
\sum_{i=1}^{5} \lambda_i = 1, \quad y \in \{0, 1\}^4, \quad \sum_{i=1}^{4} y_i = 1
\]

\[
0 \leq \lambda_1 \leq y_1
\]
\[
0 \leq \lambda_2 \leq y_1 + y_2
\]
\[
0 \leq \lambda_3 \leq y_2 + y_3
\]
\[
0 \leq \lambda_4 \leq y_3 + y_4
\]
\[
0 \leq \lambda_5 \leq y_4
\]

\[
P_i := \{ \lambda \in \Lambda^5 : \lambda_j = 0 \quad j \notin \{i, i + 1\} \} \]
What is a Formulation?

$Q = \text{LP relaxation}$

$y \in \{0, 1\}^4,$

\[
\begin{align*}
0 & \leq \lambda_1 \leq y_1 \\
0 & \leq \lambda_2 \leq y_1 + y_2 \\
0 & \leq \lambda_3 \leq y_2 + y_3 \\
0 & \leq \lambda_4 \leq y_3 + y_4 \\
0 & \leq \lambda_5 \leq y_4
\end{align*}
\]

$P_i := \{ \lambda \in \Lambda^5 : \lambda_j = 0 \mid j \notin \{i, i + 1\} \}$

Embedding Formulations
What is a Formulation?

Embedding Formulations

\[ Q = \text{LP relaxation} \]

\[ \sum_{i=1}^{5} \lambda_i = 1 \]

\[ \sum_{i=1}^{4} y_i = 1 \]

\[ (\lambda, y) \in Q \cap (\mathbb{R}^5 \times \mathbb{Z}^4) \]

\[ y = e^i \land \lambda \in P_i \]

\[ P_i := \{ \lambda \in \Lambda^5 : \lambda_j = 0 \land j \notin \{i, i+1\} \} \]
What is a Formulation?

$Q = \text{LP relaxation}$

$y \in \{0, 1\}^4,$

$\sum_{i=1}^{5} \lambda_i = 1$

$\sum_{i=1}^{4} y_i = 1$

$0 \leq \lambda_1 \leq y_1$

$0 \leq \lambda_2 \leq y_1 + y_2$

$0 \leq \lambda_3 \leq y_2 + y_3$

$0 \leq \lambda_4 \leq y_3 + y_4$

$0 \leq \lambda_5 \leq y_4$

$P_i := \{\lambda \in \Lambda^5 : \lambda_j = 0 \mid j \notin \{i, i + 1\}\}$

Unary Encoding

Embedding Formulations
What is a Formulation?

$Q = \text{LP relaxation}$

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$\sum_{i=1}^{5} \lambda_i = 1$

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$0 \leq \lambda_1 \leq y_1$

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$0 \leq \lambda_4 \leq y_3 + y_4$

$0 \leq \lambda_5 \leq y_4$

$P_i := \{ \lambda \in \Lambda^5 : \lambda_j = 0 \quad j \notin \{i, i+1\} \}$

Unary Encoding

Embedding Formulations
Alternate Meaning of 0-1 Variables

\[ Q = \text{LP relaxation} \rightarrow \sum_{i=1}^{5} \lambda_i = 1 \]

- V. and Nemhauser ’08.

0 \leq \lambda_1 + \lambda_5 \leq 1 - y_1
0 \leq \lambda_3 \leq y_1
0 \leq \lambda_4 + \lambda_5 \leq 1 - y_2
0 \leq \lambda_1 + \lambda_2 \leq y_2

\[ P_i := \{ \lambda \in \Lambda^5 : \lambda_j = 0 \quad j \notin \{i, i + 1\} \} \]

Embedding Formulations
Alternate Meaning of 0-1 Variables

\[ T_1 \quad T_2 \quad T_3 \quad T_4 \]
\[ 1 \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad 2 \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad 3 \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad 4 \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad 5 \]

\( Q = \text{LP relaxation} \longrightarrow \sum_{i=1}^{5} \lambda_i = 1 \)

\[ h^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad h^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad h^3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad h^4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

- V. and Nemhauser ’08.

\[
0 \leq \lambda_1 + \lambda_5 \leq 1 - y_1 \\
0 \leq \lambda_3 \leq y_1 \\
0 \leq \lambda_4 + \lambda_5 \leq 1 - y_2 \\
0 \leq \lambda_1 + \lambda_2 \leq y_2
\]

\[ (\lambda, y) \in Q \cap (\mathbb{R}^5 \times \mathbb{Z}^2) \]

\[ \iff \]

\[ y = h^i \land \lambda \in P_i \]

\[ P_i := \{ \lambda \in \Lambda^5 : \lambda_j = 0 \quad j \notin \{i, i+1\} \} \]
Alternate Meaning of 0-1 Variables

\[ Q = \text{LP relaxation} \quad \sum_{i=1}^{5} \lambda_i = 1 \]

\[ h^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad h^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad h^3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad h^4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

\[
\begin{align*}
0 & \leq \lambda_1 + \lambda_5 \leq 1 - y_1 \\
0 & \leq \lambda_3 \leq y_1 \\
0 & \leq \lambda_4 + \lambda_5 \leq 1 - y_2 \\
0 & \leq \lambda_1 + \lambda_2 \leq y_2
\end{align*}
\]

\[
(\lambda, y) \in Q \cap (\mathbb{R}^5 \times \mathbb{Z}^2) \quad \Leftrightarrow \quad y = h^i \land \lambda \in P_i
\]

\[ P_i := \{ \lambda \in \Lambda^5 : \lambda_j = 0 \quad j \notin \{i, i+1\} \} \]
Embedding Formulations for Union of Polyhedra

• **Non-Extended** formulation of \( \lambda \in \bigcup_{i=1}^{n} P_i \subseteq \mathbb{R}^V \):
  
  – Encoding \( H := \{h^i\}_{i=1}^{n} \subseteq \{0, 1\}^k, \quad h^i \neq h^j \)
  
  – Polyhedron \( Q \subseteq \mathbb{R}^V \times \mathbb{R}^k \), s.t.

\[
(\lambda, y) \in Q \cap (\mathbb{R}^V \times \mathbb{Z}^k) \iff y = h^i \land \lambda \in P_i
\]
Embedding Formulations for Union of Polyhedra

- **Non-Extended formulation** of $\lambda \in \bigcup_{i=1}^{n} P_i \subseteq \mathbb{R}^V$:
  - Encoding $H := \{h^i\}_{i=1}^{n} \subseteq \{0, 1\}^k$, $h^i \neq h^j$
  - Polyhedron $Q \subseteq \mathbb{R}^V \times \mathbb{R}^k$, s.t. $(\lambda, y) \in Q \cap (\mathbb{R}^V \times \mathbb{Z}^k) \iff y = h^i \land \lambda \in P_i$

- **Embedding formulation** = strongest polyhedron (ideal):
  $$Q(H) := \text{conv} \left( \bigcup_{i=1}^{n} P_i \times \{h^i\} \right)$$
Embedding Formulations for Union of Polyhedra

- **Non-Extended** formulation of $\lambda \in \bigcup_{i=1}^{n} P_i \subseteq \mathbb{R}^V$:
  
  - Encoding $H := \{ h^i \}_{i=1}^{n} \subseteq \{0, 1\}^k$, $h^i \neq h^j$
  
  - Polyhedron $Q \subseteq \mathbb{R}^V \times \mathbb{R}^k$, s.t.
    
    $$(\lambda, y) \in Q \cap (\mathbb{R}^V \times \mathbb{Z}^k) \iff y = h^i \land \lambda \in P_i$$

- **Embedding formulation** = strongest polyhedron (ideal):
  
  $$Q(H) := \text{conv} \left( \bigcup_{i=1}^{n} P_i \times \{h^i\} \right)$$

For unary encoding:

$$h^i = e^i$$
Embedding Formulation = Ideal non-Extended

$\mathbf{P}_1$ and $\mathbf{P}_2$
Embedding Formulation = Ideal non-Extended

$\left( P_1 \times \{0\} \right) \cup \left( P_2 \times \{1\} \right)$
Embedding Formulation = Ideal non-Extended

\[ Q(H) := \text{conv} \left( (P_1 \times \{0\}) \cup (P_2 \times \{1\}) \right) \]
Embedding Formulation = Ideal non-Extended

\[ Q(H) := \text{conv} \left( \left( P_1 \times \{0\} \right) \cup \left( P_2 \times \{1\} \right) \right) \]

\((x, y) \in Q(H) \cap \left( \mathbb{R}^2 \times \mathbb{Z} \right) \Rightarrow x \in P_1 \cup P_2\]
Embedding Formulation = Ideal non-Extended

\[ Q(H) := \text{conv} \left( (P_1 \times \{0\}) \cup (P_2 \times \{1\}) \right) \]

\[(x, y) \in Q(H) \cap (\mathbb{R}^2 \times \mathbb{Z}) \implies x \in P_1 \cup P_2\]

\[\text{ext}(Q(H)) \subseteq \mathbb{R}^2 \times \mathbb{Z}\]
Complexity of Family of Polyhedra $\mathcal{P} := \{P_i\}_{i=1}^n$

- Embedding complexity = smallest **ideal** formulation

$$mc(\mathcal{P}) := \min_H \{\text{size}(Q(H))\}$$

$$\text{size}(Q) := \# \text{ of facets of } Q$$
Complexity of Family of Polyhedra $\mathcal{P} := \{P_i\}_{i=1}^{n}$

- Embedding complexity = smallest **ideal** formulation

\[
mc(\mathcal{P}) := \min_H \{\text{size}(Q(H))\}
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Complexity of Family of Polyhedra $\mathcal{P} := \{P_i\}_{i=1}^{n}$

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Complexity of Family of Polyhedra $\mathcal{P} := \{P_i\}_{i=1}^n$

- **Embedding complexity** = smallest **ideal** formulation

$$mc(\mathcal{P}) := \min_H \{size(Q(H)) \}$$

- **Relaxation complexity** = smallest formulation

$$rc(\mathcal{P}) := \min_{Q,H} \{size(Q) \}$$

$$size(Q) := \# \text{ of facets of } Q$$
Complexity of Family of Polyhedra $\mathcal{P} := \{P_i\}_{i=1}^n$

- Embedding complexity = smallest **ideal** formulation

$$mc(\mathcal{P}) := \min_H \{\text{size}(Q(H))\}$$

- Relaxation complexity = smallest formulation

$$rc(\mathcal{P}) := \min_{Q,H} \{\text{size}(Q)\}$$

$$\text{size}(Q) := \# \text{ of facets of } Q$$
Summary of Results

- Lower and Upper bounds for special structures:
  - e.g. for Special Order Sets of Type 2 (SOS2) on $n$ variables
    - Embedding complexity (ideal)
      \[ 2 \lceil \log_2 n \rceil \]
      \[ n + 1 \leq \ldots \leq n + 1 + 2 \lceil \log_2 n \rceil \]
    - Relaxation complexity (non-ideal)
      \[ 2 \leq \ldots \leq 4 \]
      \[ 2 \leq \ldots \leq 5 + 2n \]

- Relation to other complexity measures
  \[ \text{hc} (\mathcal{P}) := \text{size} \left( \text{conv} \left( \bigcup_{i=1}^{n} P_i \right) \right) \]
  \[ \text{xc} (\mathcal{P}) := \min_{R} \left\{ \text{size} (R) : \text{proj}_x (R) = \text{conv} \left( \bigcup_{i=1}^{n} P_i \right) \right\} \]

- Still open questions (see V. 2015)
Why bounds? Encoding Selection Matters

\[ f(x, y) \]

Embedding Formulations
Why bounds? Encoding Selection Matters

• Size of unary formulation is:
  (Lee and Wilson ’01)

\[
\left(\frac{2\sqrt{n/2}}{\sqrt{n/2}}\right) + \left(\sqrt{n/2} + 1\right)^2
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Why bounds? Encoding Selection Matters

- **Size of unary formulation is:**
  (Lee and Wilson ’01)

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General Inequalities
**Why bounds? Encoding Selection Matters**

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  \]

  (Lee and Wilson ’01)

- **General Inequalities**

- **Variable Bounds**

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Embedding Formulations

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Why bounds? Encoding Selection Matters

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  (Lee and Wilson ’01)

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• Size of one binary formulation:
  (V. and Nemhauser ’08)
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- Size of one binary formulation:
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  \[
  4 \log_2 \sqrt{n/2} + 2 + \left( \sqrt{\frac{n}{2}} + 1 \right)^2
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Why bounds? Encoding Selection Matters

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  (Lee and Wilson ’01)

- **Size of one binary formulation:**
  
  \[
  4 \log_2 \sqrt{n/2} + 2 + \left( \sqrt{n/2} + 1 \right)^2
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  (V. and Nemhauser ’08)

- **Right embedding = significant computational advantage over alternatives (Extended, Big-M, etc.)**
Summary

• Embedding Formulations = Systematic procedure
  – Encoding can significantly affect size

• Complexity of Union of Polyhedra beyond convex hull
  – Embedding Complexity (non-extended ideal formulation)
  – Relaxation Complexity (any non-extended formulation)
  – Still open questions on relations between complexity

• More details (practical formulation construction)

• Application to facility layout problem (Huchette, Dey, V. ‘14)
  – INFORMS 2015, Philadelphia, Monday, Nov 2\textsuperscript{nd}, 13:30 - 15:00
  – MC11, 11-Franklin 1, Marriott