Extended Formulations for Quadratic Mixed Integer Programming

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SIAM Conference on Optimization, May 2014 – San Diego, California
Nonlinear MIP B&B Algorithms

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\begin{align*}
\text{max} & \quad \sum_{i=1}^{n} c_i x_i \\
\text{s.t.} & \quad g_i(x) \leq 0, \quad i \in I, \\
& \quad x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}
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Nonlinear MIP B&B Algorithms

- NLP (QCP) Based B&B

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MINLP B&B Algorithms

- NLP (QCP) Based B&B
- (Dynamic) LP Based B&B
  - Few cuts = high speed.
  - Possible slow convergence.

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- NLP (QCP) Based B&B
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- (Dynamic) LP Based B&B
- Lifted LP B&B
  - Fixed extended relaxation.
  - Mimic NLP B&B.

Mathematical formulation:

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\begin{align*}
\text{max} & \quad \sum_{i=1}^{n} c_i x_i \\
\text{s.t.} & \quad Ax + Dz \leq b, \\
& \quad g_i(x) \leq 0, \quad i \in I, \\
& \quad x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}
\end{align*}
\]
Problem 1: Classical

\[
\begin{align*}
\max_{x, y} & \quad \bar{a}y \\
\text{s.t.} & \quad \|Q^{1/2}y\|_2 \leq \sigma \\
& \quad \sum_{j=1}^{n} y_j = 1 \\
& \quad y_j \leq x_j \quad \forall j \in \{1, \ldots, n\} \\
& \quad \sum_{j=1}^{n} x_j \leq 10 \\
& \quad x \in \{0, 1\}^n \\
& \quad y \in \mathbb{R}^n_+ 
\end{align*}
\]

- \(y\) fraction of the portfolio invested in each of \(n\) assets.
- \(\bar{a}\) expected returns of assets.
- \(Q^{1/2}\) positive semidefinite square root of the covariance matrix \(Q\) of returns.
- Hold at most 10 assets.
Avg. of Solve Times [s] for $n \in \{20, 30\}$ (CPLEX v11)
Approximation of Second Order Cone by Ben-Tal and Nemirovski (Glineur).

$O(d \log(1/\varepsilon))$ variables and constraints for quality $\varepsilon$.

Problem:
- Fixed a-priori quality: no dynamic improvement.
- e.g. $\varepsilon = 0.01$ for portfolio had to be calibrated.
Dynamic Lifted Approximations
Towards a Dynamic Lifted LP

- Separable approach by Tawarmalani and Sahinidis ’05 and Hijazi et al. ’14

\[ B^n := \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} \left( x_i - \frac{1}{2} \right)^2 \leq \frac{n - 1}{4} \right\} \]

Showing \( B^n \cap \mathbb{Z}^n = \emptyset \) requires \( 2^n \) cuts.
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Extended formulation of \( B^n \):

\[
\left( x_i - \frac{1}{2} \right)^2 \leq z_i \quad \forall i \in [n]
\]

\[ \sum_{i=1}^{n} z_i \leq \frac{n - 1}{4} \]
Towards a Dynamic Lifted LP

- Separable approach by Tawarmalani and Sahinidis ’05 and Hijazi et al. ‘14

Extended Formulations

\[ B^n := \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} \left( x_i - \frac{1}{2} \right)^2 \leq \frac{n - 1}{4} \right\} \]

Extended formulation of \( B^n \):

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\left( x_i - \frac{1}{2} \right)^2 \leq z_i \quad \forall i \in [n]
\]

\[
\sum_{i=1}^{n} z_i \leq \frac{n - 1}{4}
\]

\( B^n \cap \mathbb{Z}^n = \emptyset \) with only \( 2n \) cuts on extended formulation.
Towards a Dynamic Lifted LP

Separable approach works for any set of the form:

\[ C = \left\{ x \in \mathbb{R} : \sum_{i=1}^{n} f_i(x_i) \leq 1 \right\} \]

or

\[ C = \left\{ (x, t) \in \mathbb{R} \times \mathbb{R} : \sum_{i=1}^{n} f_i(x_i) \leq t \right\} \]

for convex \( f_i : \mathbb{R} \rightarrow \mathbb{R} \)
Problem 1: Classical

\[
\begin{align*}
\max_{x,y} \quad & \bar{a}y \\
\text{s.t.} \quad & \|Q^{1/2}y\|_2 \leq \sigma \\
& \sum_{j=1}^{n} y_j = 1 \\
& y_j \leq x_j \quad \forall j \in \{1, \ldots, n\} \\
& \sum_{j=1}^{n} x_j \leq K \\
& x \in \{0, 1\}^n \\
& y \in \mathbb{R}_+^n
\end{align*}
\]

- $y$ fraction of the portfolio invested in each of $n$ assets.
- $\bar{a}$ expected returns of assets.
- $Q^{1/2}$ positive semidefinite square root of the covariance matrix $Q$ of returns.
- $K$ maximum number of assets to hold.
Problem 2: Shortfall

\[
\begin{align*}
\max_{x,y} & \quad \bar{a} y \\
\text{s.t.} & \quad ||Q^{1/2} y||_2 \leq \sigma \\
& \quad \sum_{j=1}^{n} y_j = 1 \\
& \quad y_j \leq x_j \quad \forall j \in \{1, \ldots, n\} \\
& \quad \sum_{j=1}^{n} x_j \leq K \\
& \quad x \in \{0, 1\}^n \\
& \quad y \in \mathbb{R}_+^n
\end{align*}
\]

- \( \bar{a} \) expected returns of assets.
- \( Q^{1/2} \) positive semidefinite square root of the covariance matrix \( Q \) of returns.
- \( K \) maximum number of assets to hold.
- \( \sigma \) fraction of the portfolio invested in each of \( n \) assets.
Problem 2 : Shortfall

\[
\begin{align*}
\max_{x,y} & \quad \bar{a} y \\
\text{s.t.} & \quad \|Q^{1/2} y\|_2 \leq \frac{\bar{a} y - W_i^{low}}{\Phi^{-1}(\eta_i)} \quad i \in \{1, 2\} \\
& \quad \sum_{j=1}^{n} y_j = 1 \\
& \quad y_j \leq x_j \quad \forall j \in \{1, \ldots, n\} \\
& \quad \sum_{j=1}^{n} x_j \leq K \\
& \quad x \in \{0, 1\}^n \\
& \quad y \in \mathbb{R}_+^n
\end{align*}
\]

- \( y \) fraction of the portfolio invested in each of \( n \) assets.
- \( \bar{a} \) expected returns of assets.
- \( Q^{1/2} \) positive semidefinite square root of the covariance matrix \( Q \) of returns.
- \( K \) maximum number of assets to hold.
- Approximation of \( \text{Prob}(\bar{a} y \geq W_i^{low}) \geq \eta_i \)
\[ L^n = \{(x, t) \in \mathbb{R} \times \mathbb{R} : \|x\| \leq t\} \]

Extended formulation of \(L^n = \)

homogenization of \(B^n\) formulation:

\[ x_i^2 \leq z_i \cdot t \quad \forall i \in [n] \]

\[ \sum_{i=1}^{n} z_i \leq t \]
Extended Formulation for SOCP

\[ L^n = \{ (x, t) \in \mathbb{R} \times \mathbb{R} : \|x\| \leq t \} \]

Extended formulation of \( L^n \) = homogenization of \( B^n \) formulation:

\[ x_i^2 \leq z_i \cdot t \quad \forall i \in [n] \]

\[ \sum_{i=1}^{n} z_i \leq t \]

Rotated SOCP cone
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  - Mimic NLP B&B.

- Lifted LP B&B
  - LP B&B on extended form.

- Dynamic Lifted LP B&B
  - LP B&B on extended form.
Extended Formulations

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Maximize \( \sum_{i=1}^{n} c_i x_i \)
subject to
\[ g_i(x, z) \leq 0, i \in I, \]
\[ g_i(x) \leq 0, i \in I, \]
\[ x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \]
Computational Results

- Averages over 20 instances:
  - Classical and Shortfall. 40, 50 and 60 stocks.

- Solvers:
  - CPLEX/Gurobi QCP-BB on original formulation.
  - Lifted LP: Implemented in JuMP using CPLEX’s branch, incumbent and heuristic callback.
  - CPLEX/Gurobi LP-BB on extended “separable” reformulation.
Computational Results

LiftedLP v/s QCP: Classical

- LiftedLP
- CPLEX QCP
- Gurobi QCP
Computational Results

LiftedLP v/s QCP: Shortfall

- LiftedLP
- CPLEX QCP
- Gurobi QCP

Graph showing the comparison between LiftedLP, CPLEX QCP, and Gurobi QCP for different values.
LiftedLP v/s Dynamic : Classical

- LiftedLP
- CPLEX SEP-LP
- Gurobi SEP-LP
Computational Results

LiftedLP v/s Dynamic : Shortfall

- LiftedLP
- CPLEX SEP-LP
- Gurobi SEP-LP
Summary

- Lifted LP: 200 lines of JuMP code in a weekend.
- Developed by ORC students Iain Dunning, Joey Huchette and Miles Lubin
- https://github.com/JuliaOpt/JuMP.jl
- Poster at MIP 2014. OSU, July 21st
- Talk at INFORMS. San Francisco, November

- Dynamic Lifted LP:
  - Comparable performance with simple reformulation.