A Constructive Characterization of the Split Closure of a Mixed Integer Linear Program

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What is the Split Closure

- **Split Cuts:**
  - Valid Inequalities “equivalent” to Intersection Cuts, Mixed Integer Gomory Cuts and MIR Cuts.
  - Special case of Balas’s Disjunctive Cuts.

- **Closure:**
  - Obtained by adding all cuts in a class.
  - Class could have infinite number of cuts, so closures are not immediately polyhedrons.
  - Example: Chvátal Closure (Is a polyhedron).
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History and Motivation

- **History:**
  - Split Cuts were introduced by [Cook, et. al. 1990].
  - Split Closure is a polyhedron
    [Cook, et. al. 1990, Andersen, et. al. 2005].
    Non-constructive proofs.
  - The Split Closure has recently been studied by

- **Motivation of Constructive Characterization:**
  - Algorithm to generate Split Closure? (Naive).
  - Helps understand Split Cuts better.
  - For fixed dimension. Is the number of inequalities defining
    the Split Closure polynomial in the size of the input? (Open
    even for two inequalities in $\mathbb{R}^2$).
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6. Valid Inequalities for Mixed-Integer Sets

**Figure 6.2**

**Proposition 6.3.** Given the two valid inequalities (6.3) for $T$, it follows that (6.4) is also valid for $T$. 
Feasible set:

- \( P := \{ x \in \mathbb{R}^n : a_i^T x \leq b_i \quad \forall i \in M \} \)
- \( P_I := \{ x \in P : x_j \in \mathbb{Z} \quad \forall j \in N_I \} \) for \( N_I \subseteq \{1, \ldots, n\} \)
Feasible Set of a (Mixed) Integer Linear Program and Natural Relaxations

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Relaxations:

- \( P \), LP Relaxation
  \( P(B) := \{ x \in \mathbb{R}^n : a_i^T x \leq b_i \quad \forall \ i \in B \} \) for \( B \in \mathcal{B} := \{ B \subseteq M : |B| = n, \ \{a_i\}_{i \in B} \ l.i. \} \)
- Basic or Conic Relaxation
  \( \{ x \in P(B) : x_j \in \mathbb{Z} \quad \forall \ j \in N_I \} \) is a relaxation of \( P_I \).
- \( x(B) \) unique solution to \( a_i^T x = b_i \quad \forall \ i \in B \)
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Split Cuts are Constructed from Valid Split Disjunctions

For $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ divide $\mathbb{R}^n$ into:

- $F^l := \{x \in \mathbb{R}^n : \pi^T x \leq \pi_0\}$
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A split cut for \(D(\pi, \pi_0)\) and \(P\) is an inequality valid for:

- \(P^l \cup P^g\)
- \(\text{conv}(P^l_{(\pi, \pi_0)} \cup P^g_{(\pi, \pi_0)})\)
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Valid Splits don’t Cut Integer Feasible Points

For fixed $N_I$ we are interested in $(\pi, \pi_0)$ such that, for any $P$:

- $P_I \subseteq F^l \cup F^g \subseteq \mathbb{R}^n$
Valid Splits don’t Cut Integer Feasible Points

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so we study

$$\Pi(N_I) := \{(\pi, \pi_0) \in (\mathbb{Z}^n \setminus \{0\}) \times \mathbb{Z} : \pi_j = 0, j \notin N_I\}$$
The Split Closure is the *Polyhedron* Formed by All Split Cuts

The *split closure* [Cook, et. al. 1990] of $P_I$ is

$$SC := \bigcap_{(\pi, \pi_0) \in \Pi(N_I)} \text{conv}(P^l(\pi, \pi_0) \cup P^g(\pi, \pi_0)).$$

**Theorem**

[Cook, et. al. 1990] *SC is a polyhedron*
For basis $B \in \mathcal{B}$ let

- $P(B)^l := \{ x \in P(B) : \pi^T x \leq \pi_0 \}$
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and

$$ SC(B) := \bigcap_{(\pi,\pi_0) \in \Pi(N_I)} \text{conv}(P(B)^l_{(\pi,\pi_0)} \cup P(B)^g_{(\pi,\pi_0)}). $$
Sufficient to Study Split Cuts for Basic Relaxations

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**Theorem**

[Andersen, et. al. 2005] $SC = \bigcap_{B \in \mathcal{B}} SC(B)$

**Theorem**

[Andersen, et. al. 2005] $SC(B)$ is a polyhedron for all $B \in \mathcal{B}$. Hence $SC$ is a polyhedron.
Farkas’s Lemma Can be Used to Characterize Split Cuts

Let $P = P(B) = \{ x \in \mathbb{R}^n : Bx \leq b \}$, for $B \in \mathbb{Q}^{n \times n}$, rank$(B) = n$
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- For $(\pi, \pi_0) \in \Pi(N_I)$ such that $\pi^T x(B) \in (\pi_0, \pi_0 + 1)$ let
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Split cut $\delta^T x \leq \delta_0$ is valid for $P^l$ and $P^g$:

- F.L. for $P^l$: $\exists (\mu^l_0, \mu^l) \in \mathbb{R}_+ \times \mathbb{R}^n_+$ s.t.
  - $\delta = B^T \mu^l + \mu^l_0 \pi$
  - $\delta_0 b^T \mu^l + \mu^l_0 \pi_0$

- F.L. for $P^g$: $\exists (\mu^g_0, \mu^g) \in \mathbb{R}_+ \times \mathbb{R}^n_+$ s.t.
  - $\delta = B^T \mu^g - \mu^g_0 \pi$
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For \( (\pi, \pi_0) \in \Pi(N_l) \) such that \( \pi^T x(B) \in (\pi_0, \pi_0 + 1) \) let
\[
\begin{align*}
P^l &:= \{ x \in P : \pi^T x \leq \pi_0 \} \\
P^g &:= \{ x \in P : \pi^T x \geq \pi_0 + 1 \}
\end{align*}
\]

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\[
\begin{align*}
\delta &= B^T \mu^l + \mu_0^l \pi \\
\delta_0 &\geq b^T \mu^l + \mu_0^l \pi_0
\end{align*}
\]

**F.L. for** \( P^g \): \( \exists (\mu^g_0, \mu^g) \in \mathbb{R}_+ \times \mathbb{R}_+^n \) s.t.
\[
\begin{align*}
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\delta_0 &\geq b^T \mu^g - \mu_0^g (\pi_0 + 1)
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    - $\delta_0 = b^T \mu^g - \mu^g_0 (\pi_0 + 1)$
Characterization

Lattices

Polyhedrality
[Andersen, et. al. 2005, Balas and Perregaard, 2003, Caprara and Letchford, 2003] All non-dominated valid inequalities for $\text{conv}(P_l^{(\pi,\pi_0)} \cup P_g^{(\pi,\pi_0)})$ are of the form $\delta^T x \leq \delta_0$ where

$$
\delta = B^T \mu^l + \mu^l_0 \pi = B^T \mu^g - \mu^g_0 \pi
$$

$$
\delta_0 = b^T \mu^l + \mu^l_0 \pi_0 = b^T \mu^g - \mu^g_0 (\pi_0 + 1)
$$

for $\mu^l_0, \mu^g_0 \in \mathbb{R}_+$ and $\mu^l, \mu^g \in \mathbb{R}_n^+$ solutions to

$$
B^T \mu^g - B^T \mu^l = \pi
$$

$$
b^T \mu^g - b^T \mu^l = \pi_0 + \mu^g_0
$$

$$
\mu^l_0 + \mu^g_0 = 1, \quad \mu^g_0 \in (0, 1), \quad \mu^l_i \cdot \mu^g_i = 0
$$
\[ B^T \mu^g - B^T \mu^l = \pi \]
\[ b^T \mu^g - b^T \mu^l = \pi_0 + \mu_0^g \]
\[ \mu^l, \mu^g \in \mathbb{R}_+^n, \quad \mu_i^l \cdot \mu_i^g = 0 \]
\[ \mu_0^g \in (0, 1), \quad \pi_0 \in \mathbb{Z} \]
\[ B^T \mu^g - B^T \mu^l = \pi \]
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\[ B^T \mu = \pi \]
\[ b^T \mu = \pi_0 + \mu_0^g \]
\[ \mu \in \mathbb{R}^n \]
\[ \mu_0^g \in (0, 1), \quad \pi_0 \in \mathbb{Z} \]

\[ \mu_i^l = (\mu_i)^- := \max\{-\mu_i, 0\} \]
\[ B^T \mu = \pi \]
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\[ B^T \mu = \pi \]
\[ \lfloor b^T \mu \rfloor = \pi_0 \]
\[ \mu \in \mathbb{R}^n \]
\[ \mu^T b \notin \mathbb{Z} \]

\[ \mu_i^l = (\mu_i)^- := \max\{-\mu_i, 0\}, \quad \mu_0^g = f(b^T \mu) := b^T \mu - \lfloor b^T \mu \rfloor \]
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\[ Bx(B) = b \]
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\[ \pi_0 < \pi^T x(B) < \pi_0 + 1 \]
\[
B^T \mu = \pi \\
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\( \mu_i^l = (\mu_i)^- := \max\{-\mu_i, 0\}, \quad \mu_0^g = f(b^T \mu), \quad \mu_0^l = 1 - \mu_0^g \)

\( \delta = B^T \mu^l + \mu_0^l \pi \)
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\[ \delta_0 = b^T \mu^l + \mu^l_0 \pi_0 \]
\begin{align*}
B^T \mu &= \pi \\
\lfloor b^T \mu \rfloor &= \pi_0 \\
\mu &\in \mathbb{R}^n \\
\mu^T b &\notin \mathbb{Z}
\end{align*}

\begin{align*}
\mu_i^- &= (\mu_i)^- := \max\{-\mu_i, 0\}, \quad \mu_0^l = 1 - f(b^T \mu) \\
\delta &= B^T \mu^- + (1 - f(b^T \mu))\pi \\
\delta_0 &= b^T \mu^- + (1 - f(b^T \mu))\pi_0
\end{align*}
\[ B^T \mu = \pi \]
\[ \lfloor b^T \mu \rfloor = \pi_0 \]
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\[\delta = B^T \mu^- + (1 - f(b^T \mu))B^T \mu\]
\[\delta_0 = b^T \mu^- + (1 - f(b^T \mu))\lfloor b^T \mu \rfloor\]
Proposition

$$\text{conv}(P^l_{(\pi, \pi_0)} \cup P^g_{(\pi, \pi_0)}) = \{x \in P : \delta^T x \leq \delta_0\}$$

where $\delta(\mu)^T x \leq \delta_0(\mu)$ is

$$(\mu^-)^T (Bx - b) + (1 - f(\mu^T b))(\mu^T Bx - \lfloor \mu^T b \rfloor) \leq 0$$

for $\mu$ unique solution (if it exists) to

$$B^T \mu = \pi \quad \mu \in \mathbb{R}^n$$

$$\mu^T b \notin \mathbb{Z} \quad \pi_0 = \lfloor \mu^T b \rfloor$$

$$(y^- = \max\{-y, 0\}, f(y) = y - \lfloor y \rfloor \text{ and operations over vectors are component wise})$$
What Multipliers Induce Valid Split Disjunctions?

- We have
  \[ \Pi(N_I) := \{(\pi, \pi_0) \in (\mathbb{Z}^n \setminus \{0\}) \times \mathbb{Z} : \pi_j = 0, j \notin N_I\} \] and

  \[ B^T \mu = \pi \quad \mu \in \mathbb{R}^r \]

  \[ \mu^T b \notin \mathbb{Z} \quad \pi_0 = \lfloor \mu^T b \rfloor \]

- Let \( B = [B_I B_C] \) for \( B_I \in \mathbb{R}^{n \times |N_I|} \) and \( B_C \in \mathbb{R}^{n \times (n-|N_I|)} \) corresponding to the integer and continuous variables of \( P_I \). Multipliers that induce valid split disjunctions are

  \[ \mathcal{L}(B) := \{\mu \in \mathbb{R}^n : B_I^T \mu \in \mathbb{Z}^{|N_I|}, \ B_C^T \mu = 0\} \]
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Valid Split Disjunctions are Related to Integer Lattices

For \( \{v^i\}_{i=1}^r \subseteq \mathbb{R}^n \) l.i. a lattice is

\[ \mathcal{L} := \{ \mu \in \mathbb{R}^n : \mu = \sum_{i=1}^r k_i v^i \quad k_i \in \mathbb{Z} \} \]

\( \mathcal{L}(B) \) is a lattice,

\[ [\mu^-]^T (Bx - b) + (1 - f(\mu^T b)) (\mu^T Bx - [\mu^T b]) \leq 0 \]

is valid for \( P_I \) and cuts \( x(B) \).

[Köppe and Weismantel, 2004].

Every \( \mu \in \mathcal{L}(B) \) s.t. \( \mu^T b \notin \mathbb{Z} \) induces a valid split disjunction.

[Bertsimas and Weismantel, 2005].
Valid Split Disjunctions are Related to Integer Lattices

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Valid Split Disjunctions are Related to Integer Lattices

- For \( \{v^i\}_{i=1}^r \subseteq \mathbb{R}^n \) l.i. a lattice is
  \[
  L := \{ \mu \in \mathbb{R}^n : \mu = \sum_{i=1}^{r} k_i v^i \quad k_i \in \mathbb{Z} \}
  \]

- \( L(B) \) is a lattice,
  \[
  \lfloor \mu^T \rfloor (Bx - b) + (1 - f(\mu^T b))(\mu^T Bx - \lfloor \mu^T b \rfloor) \leq 0
  \]
  is valid for \( P_I \) and cuts \( x(B) \).
  [Köppe and Weismantel, 2004].

- Every \( \mu \in L(B) \) s.t. \( \mu^T b \notin \mathbb{Z} \) induces a valid split disjunction.
  [Bertsimas and Weismantel, 2005].
Proposition

\[ SC(B) = \bigcap_{\substack{\mu \in \mathcal{L}(B) \\ \mu^T b \notin \mathbb{Z}}} \{ x \in P(B) : \delta(\mu)^T x \leq \delta_0(\mu) \}. \]
Proposition

\[ SC(B) = \bigcap_{\mu \in \mathcal{L}(B), \mu^T b \notin \mathbb{Z}} \{ x \in P(B) : \delta(\mu)^T x \leq \delta_0(\mu) \}. \]

Proposition

For \( \mu \in \mathcal{L}(B) \) s.t \( \mu^T b \notin \mathbb{Z} \) split cut

\[
(\mu^-)^T (Bx - b) + (1 - f(\mu^T b))(\mu^T Bx - \lfloor \mu^T b \rfloor) \leq 0
\]

dominates

\[
[\mu^-]^T (Bx - b) + (1 - f(\mu^T b))(\mu^T Bx - \lfloor \mu^T b \rfloor) \leq 0
\]
Studying $\mathcal{L}(B)$ in Each Orthant Decomposes $SC(B)$ to the Intersection of a *Finite* Number of Sets

For $\sigma \in \{0, 1\}^n$ let

$$\mathcal{L}(B, \sigma) := \{ \mu \in \mathcal{L}(B) : (-1)^{\sigma_i} \mu_i \geq 0, \ \forall i \in \{1, \ldots, n\} \}$$

so that

$$SC(B) = \bigcap_{\sigma \in \{0, 1\}^n} SC(B, \sigma)$$

where

$$SC(B, \sigma) = \bigcap_{\mu \in \mathcal{L}(B, \sigma) \ \mu^T b \notin \mathbb{Z}} \{ x \in P(B) : \delta(\mu)^T x \leq \delta_0(\mu) \}$$
Studying $\mathcal{L}(B, \sigma)$ Allows Detecting Dominated Cuts

Lemma

Let $\sigma \in \{0, 1\}^n$ and let $\mu \in \mathcal{L}(B, \sigma)$ with $\mu = \alpha + \beta$ for $\alpha, \beta \in \mathcal{L}(B, \sigma)$ such that $\beta^T b \in \mathbb{Z}$. Then $\delta(\mu)^T x \leq \delta_0(\mu)$ is dominated by $\delta(\alpha)^T x \leq \delta_0(\alpha)$ in $P(B)$.

Proof.

Uses the fact that for $\alpha, \beta$ in the same orthant $|\alpha_i + \beta_i| = |\alpha_i| + |\beta_i|$ for all $i \in \{1, \ldots, n\}$ and the following alternative characterization of split cuts

$$|\bar{\mu}|^T (\bar{B}x - \bar{b}) + (1 - 2f(\bar{\mu}^T \bar{b}))(\bar{\mu}^T \bar{B}x - \lfloor \bar{\mu}^T \bar{b} \rfloor) + f(\bar{\mu}^T \bar{b}) \leq 0$$
A Finite Integral Generating Set (FIGS) of $\mathcal{L}(B, \sigma)$ Induces a Finite Subset of $\mathcal{L}(B, \sigma)$

- Let $\{v^i\}_{i \in \mathcal{V}(\sigma)} \subseteq \mathcal{L}(B, \sigma)$ be a (FIGS), i.e. a finite set such that

  $$\mathcal{L}(B, \sigma) = \{\mu \in \mathbb{R}^r : \mu = \sum_{i \in \mathcal{V}(\sigma)} k_i v^i \quad k_i \in \mathbb{Z}_+\}$$

- We want $\mu^T b \not\in \mathbb{Z}$, so for $i \in \mathcal{V}(\sigma)$ let

  $$m_i = \min\{m \in \mathbb{Z}_+ \setminus \{0\} : m b^T v^i \in \mathbb{Z}\}$$

and define the following finite subset of $\mathcal{L}(B, \sigma)$.

  $$\mathcal{L}^0(B, \sigma) := \{\mu \in \mathcal{L}(B, \sigma) : \mu = \sum_{i \in \mathcal{V}(\sigma)} r_i v^i, r_i \in \{0, \ldots, m_i-1\}\}$$
A Finite Integral Generating Set (FIGS) of $\mathcal{L}(B, \sigma)$ Induces a Finite Subset of $\mathcal{L}(B, \sigma)$

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Proving the Polyhedrality of $SC(B, \sigma)$ Yields the Polyhedrality of $SC$

**Theorem**

$SC(B, \sigma)$ the polyhedron given by

$$SC(B, \sigma) = \bigcap_{\mu \in \mathcal{L}^0(B, \sigma)} \{x \in P(B) : \delta(\mu)^T x \leq \delta_0(\mu)\}$$

**Corollary**

$SC(B)$ is a polyhedron for all $B \in \mathcal{B}$. $SC$ is a polyhedron.
Goal: For $\mu \in \mathcal{L}(B, \sigma)$, $\delta(\mu)^T x \leq \delta_0(\mu)$ is dominated by $\delta(\alpha)^T x \leq \delta_0(\alpha)$ for some $\alpha \in \mathcal{L}^0(B, \sigma)$.

How:
- For $\mu \in \mathcal{L}(B, \sigma)$ show that $\mu = \alpha + \beta$ for $\alpha, \beta$ such that:
  - $\alpha \in \mathcal{L}^0(B, \sigma), \beta \in \mathcal{L}(B, \sigma)$
  - $\beta^T b \in \mathbb{Z}$
- Use Lemma.
**Proof Idea.**

- **Goal:** For $\mu \in \mathcal{L}(B, \sigma)$, $\delta(\mu)^T x \leq \delta_0(\mu)$ is dominated by $\delta(\alpha)^T x \leq \delta_0(\alpha)$ for some $\alpha \in \mathcal{L}^0(B, \sigma)$.

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    - $\beta^T b \in \mathbb{Z}$
  - Use Lemma.
Proof of Theorem.

Let \( \{v^i\}_{i \in V(\sigma)} \) be a FIGS for \( \mathcal{L}(B, \sigma) \) and let \( \{k_i\}_{i \in V(\sigma)} \subseteq \mathbb{Z}_+ \) be such that

\[
\mu = \sum_{i \in V(\sigma)} k_i v^i.
\]
Proof of Theorem.

Let \( \{v^i\}_{i \in \mathcal{V}(\sigma)} \) be a FIGS for \( \mathcal{L}(B, \sigma) \) and let \( \{k_i\}_{i \in \mathcal{V}(\sigma)} \subseteq \mathbb{Z}_+ \) be such that

\[
\mu = \sum_{i \in \mathcal{V}(\sigma)} k_i v^i.
\]

For each \( i \in \mathcal{V}(\sigma) \) we have

\[
k_i = n_i m_i + r_i
\]

for some \( n_i, r_i \in \mathbb{Z}_+ \), \( 0 \leq r_i < m_i \). Let

\[
\alpha = \sum_{i \in \mathcal{V}(\sigma)} r_i v^i \quad \text{and} \quad \beta = \sum_{i \in \mathcal{V}(\sigma)} n_i m_i v^i
\]

We have \( \alpha \in \mathcal{L}^0(B, \sigma) \) and, as \( m_i \) is such that \( m_i b^T v^i \in \mathbb{Z} \) we have \( b^T \beta \in \mathbb{Z} \). \( \square \)
The proof of the Theorem gives a way of enumerating the inequalities of $SC(B, \sigma)$, $SC(B)$ and $SC$:

- Not practical for anything but toy problems.
- There is redundancy in the enumeration for $SC$ and $SC(B)$.
- There is also redundancy in the enumeration of $SC(B, \sigma)$. In fact we can reduce $\mathcal{L}^0(B, \sigma)$ to

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\mathcal{L}^0(B, \sigma) := \{ \mu \in \mathcal{L}(B, \sigma) : \mu = \sum_{i \in \mathcal{V}(\sigma)} r_i v^i, r_i \in \{0, \ldots, m_i - 1\} \}
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and $\{r_i\}_{i \in \mathcal{V}(\sigma)}$ are relatively prime

[Dash et. al. 2008] also give a constructive characterization with similar properties.
Final Remarks

- The proof of the Theorem gives a way of enumerating the inequalities of \( SC(B, \sigma) \), \( SC(B) \) and \( SC \):
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