A Constructive Characterization of the Split Closure of a Mixed Integer Linear Program

Juan Pablo Vielma

School of Industrial and Systems Engineering
Georgia Institute of Technology

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Outline

1. Introduction
2. Characterization
3. Lattices
4. Polyhedrality
History and Motivation

History:
- Split Cuts were introduced by [Cook, et. al. 1990]. Special case of Balas’s Disjunctive Cuts. “Equivalent” Intersection Cuts, Mixed Integer Gomory Cuts and MIR Cuts.
- The Split Closure is obtained by applying all split cuts.
- Split Closure is a polyhedron [Cook, et. al. 1990, Andersen, et. al. 2005].
- Non-constructive proofs.
- The Split Closure has recently been studied by [Balas and Saxena, 2005] and by [Dash et. al. 2005].

Motivation of Constructive Characterization:
- Algorithm to generate Split Closure? (Naive).
- Helps understand Split Cuts better.
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  - The Split Closure is obtained by applying all split cuts.
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Feasible Set of a (Mixed) Integer Linear Program and Natural Relaxations

Feasible set:

- \( P := \{ x \in \mathbb{R}^n : a_i^T x \leq b_i \quad \forall i \in M \} \)
- \( P_I := \{ x \in P : x_j \in \mathbb{Z} \quad \forall j \in N_I \} \) for \( N_I \subseteq \{1, \ldots, n\} \)
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Relaxations:

- \( P, \) LP Relaxation
  - \( P(B) := \{x \in \mathbb{R}^n : a_i^T x \leq b_i \quad \forall i \in B\} \) for \( B \in \mathcal{B} := \{B \subseteq M : |B| = n, \ \{a_i\}_{i \in B} \text{ l.i.}\} \)
  - Basic or Conic Relaxation
  - \( x(B) \) unique solution to \( a_i^T x = b_i \quad \forall i \in B \)
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Basic or Conic Relaxation

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Split Cuts are Constructed from Valid Split Disjunctions

For $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ let:

- $F_{D(\pi,\pi_0)}^l := \{x \in \mathbb{R}^n : \pi^T x \leq \pi_0\}$
- $F_{D(\pi,\pi_0)}^g := \{x \in \mathbb{R}^n : \pi^T x \geq \pi_0 + 1\}$
- $F_{D(\pi,\pi_0)} := F_{D(\pi,\pi_0)}^l \cup F_{D(\pi,\pi_0)}^g$
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For \((\pi, \pi_0) \in \mathbb{Z}^{n+1}\) let:

- \(F^l_D(\pi, \pi_0) := \{x \in \mathbb{R}^n : \pi^T x \leq \pi_0\}\)
- \(F^g_D(\pi, \pi_0) := \{x \in \mathbb{R}^n : \pi^T x \geq \pi_0 + 1\}\)
- \(F_D(\pi, \pi_0) := F^l_D(\pi, \pi_0) \cup F^g_D(\pi, \pi_0)\)

A split cut for \(D(\pi, \pi_0)\) and \(P\) is an inequality valid for:

- \(P \cap F_D(\pi, \pi_0)\)
- \(\text{conv}(P \cap F_D(\pi, \pi_0))\)
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A split cut for \(D(\pi, \pi_0)\) and \(P\) is an inequality valid for:

- \(P \cap F_{D(\pi, \pi_0)}\)
- \(\text{conv}(P \cap F_{D(\pi, \pi_0)})\)
Valid Split Disjunctions don’t Cut Integer Feasible Points

For fixed $N_I$ we are interested in $D(\pi, \pi_0)$ such that, for any $P$:

- $P_I \subseteq F_D(\pi, \pi_0) \subsetneq \mathbb{R}^n$
Valid Split Disjunctions don’t Cut Integer Feasible Points

For fixed $N_I$ we are interested in $D(\pi, \pi_0)$ such that, for any $P$:

- $P_I \subseteq F_{D(\pi, \pi_0)} \not\subseteq \mathbb{R}^n$

so we study

$$\Pi(N_I) := \{ (\pi, \pi_0) \in (\mathbb{Z}^n \setminus \{0\}) \times \mathbb{Z} : \pi_j = 0, j \notin N_I \}$$
The Split Closure is the *Polyhedron* Formed by All Split Cuts

The *split closure* [Cook, et. al. 1990] of $P_I$ is

$$SC := \bigcap_{(\pi, \pi_0) \in \Pi(N_I)} \text{conv}(P \cap F_D(\pi, \pi_0)).$$

**Theorem**

[Cook, et. al. 1990] $SC$ is a polyhedron
Sufficient to Study Split Cuts for Basic Relaxations

For $B \in \mathcal{B}$ let

$$SC(B) := \bigcap_{(\pi, \pi_0) \in \Pi(N_f)} \text{conv}(P(B) \cap F_D(\pi, \pi_0)).$$
Sufficient to Study Split Cuts for Basic Relaxations

For $B \in \mathcal{B}$ let

$$SC(B) := \bigcap_{(\pi,\pi_0) \in \Pi(N_f)} \text{conv} \left( P(B) \cap F_D(\pi,\pi_0) \right).$$

**Theorem**

[Andersen, et. al. 2005] $SC = \bigcap_{B \in \mathcal{B}} SC(B)$

**Theorem**

[Andersen, et. al. 2005] $SC(B)$ is a polyhedron for all $B \in \mathcal{B}$. Hence $SC$ is a polyhedron.

- Let $P = P(B) = \{x \in \mathbb{R}^n : Bx \leq b\}$, for $B \in \mathbb{Q}^{n \times n}$, $\text{rank}(B) = n$
Proposition

[Andersen, et. al. 2005, Balas and Perregaard, 2003, Caprara and Letchford, 2003] All non-dominated valid inequalities for $\text{conv}(P \cap F_D(\pi, \pi_0))$ are of the form $\delta^T x \leq \delta_0$ where

\[
\delta = B^T \mu^l + \mu_0^l \pi = B^T \mu^g - \mu_0^g \pi \\
\delta_0 = b^T \mu^l + \mu_0^l \pi_0 = b^T \mu^g - \mu_0^g (\pi_0 + 1)
\]

for $\mu_0^l, \mu_0^g \in \mathbb{R}_+$ and $\mu^l, \mu^g \in \mathbb{R}_+^m$ solutions to

\[
B^T \mu^g - B^T \mu^l = \pi \\
b^T \mu^g - b^T \mu^l = \pi_0 + \mu_0^g
\]

$\mu_0^l + \mu_0^g = 1$, $\mu_0^g \in (0, 1)$, $\mu_i^l \cdot \mu_i^g = 0$
Proposition

\[
\text{conv}(P \cap F_{D(\pi, \pi_0)}) = \{x \in P : \delta^T x \leq \delta_0\}
\]

where \( \delta(\mu)^T x \leq \delta_0(\mu) \) is defined equivalent to

\[
(\mu^-)^T (Bx - b) + (1 - f(\mu^T b))(\mu^T Bx - \lfloor \mu^T b \rfloor) \leq 0
\]

for \( \mu \) unique solution (if it exists) to

\[
B^T \mu = \pi \quad \mu \in \mathbb{R}^r
\]

\[
\mu^T b \notin \mathbb{Z} \quad \pi_0 = \lfloor \mu^T b \rfloor
\]

\((y^- = \max\{-y, 0\}, f(y) = y - \lfloor y \rfloor\) and operations over vectors are componentwise\)
What Multipliers Induce Valid Split Disjunctions?

We have
\[ \Pi(N_I) := \{ (\pi, \pi_0) \in (\mathbb{Z}^n \setminus \{0\}) \times \mathbb{Z} : \pi_j = 0, j \notin N_I \} \] and
\[
B^T \mu = \pi \\
\mu^T b \notin \mathbb{Z} \\
\pi_0 = \lfloor \mu^T b \rfloor
\]

Let \( B = [B_I B_C] \) for \( B_I \in \mathbb{R}^{n \times |N_I|} \) and \( B_C \in \mathbb{R}^{n \times (n - |N_I|)} \) corresponding to the integer and continuous variables of \( P_I \). Multipliers that induce valid split disjunctions are
\[ \mathcal{L}(B) := \{ \mu \in \mathbb{R}^n : B_I^T \mu \in \mathbb{Z}^{N_I}, B_C^T \mu = 0 \} \]
What Multipliers Induce Valid Split Disjunctions?

- We have
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  and
  \[ B^T \mu = \pi \quad \mu \in \mathbb{R}^r \]
  \[ \mu^T b \notin \mathbb{Z} \quad \pi_0 = \lfloor \mu^T b \rfloor \]

- Let \( B = [B_IB_C] \) for \( B_I \in \mathbb{R}^{n \times |N_I|} \) and \( B_C \in \mathbb{R}^{n \times (n-|N_I|)} \) corresponding to the integer and continuous variables of \( P_I \). Multipliers that induce valid split disjunctions are
  \[ \mathcal{L}(B) := \{\mu \in \mathbb{R}^n : B_I^T \mu \in \mathbb{Z}^{|N_I|}, \quad B_C^T \mu = 0\} \]
Valid Split Disjunctions are Related to Integer Lattices

- For \( \{v^i\}_{i=1}^r \subseteq \mathbb{R}^n \)
  - A lattice is
    \[
    \mathcal{L} := \{ \mu \in \mathbb{R}^n : \mu = \sum_{i=1}^r k_i v^i, \quad k_i \in \mathbb{Z} \} \]

- \( \mathcal{L}(B) \) is a lattice,
  \[
  [\mu^-]^T (Bx-b) + (1-f(\mu^T b)) (\mu^T Bx - [\mu^T b]) \leq 0
  \]
  is valid for \( P_I \) and cuts \( x(B) \).
  [Köppe and Weismantel, 2004].

- Every \( \mu \in \mathcal{L}(B) \) s.t. \( \mu^T b \notin \mathbb{Z} \) induces a valid split disjunction.
  [Bertsimas and Weismantel, 2005].
Valid Split Disjunctions are Related to Integer Lattices

- For \( \{v^i\}_{i=1}^r \subseteq \mathbb{R}^n \) i.e. a lattice is

\[
\mathcal{L} := \{ \mu \in \mathbb{R}^n : \mu = \sum_{i=1}^{r} k_i v^i \hspace{0.5em} k_i \in \mathbb{Z} \}
\]

- \( \mathcal{L}(B) \) is a lattice,

\[
[\mu^+]^T (Bx - b) + (1 - f(\mu^T b)) (\mu^T Bx - [\mu^T b]) \leq 0
\]

is valid for \( P_I \) and cuts \( x(B) \).
[Köppe and Weismantel, 2004].

- Every \( \mu \in \mathcal{L}(B) \) s.t. \( \mu^T b \notin \mathbb{Z} \) induces a valid split disjunction.
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Valid Split Disjunctions are Related to Integer Lattices

- For \( \{v^i\}_{i=1}^r \subseteq \mathbb{R}^n \) i.i. a lattice is
  \[
  \mathcal{L} := \{\mu \in \mathbb{R}^n : \mu = \sum_{i=1}^r k_i v^i, k_i \in \mathbb{Z}\}
  \]

- \( \mathcal{L}(B) \) is a lattice,
  \[
  [\mu^-]^T (Bx - b) + (1 - f(\mu^T b))(\mu^T Bx - [\mu^T b]) \leq 0
  \]
  is valid for \( P_1 \) and cuts \( x(B) \).
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- Every \( \mu \in \mathcal{L}(B) \) s.t. \( \mu^T b \notin \mathbb{Z} \) induces a valid split disjunction.
  [Bertsimas and Weismantel, 2005].
Proposition

\[ SC(B) = \bigcap_{\mu \in \mathcal{L}(B) \atop \mu^T b \notin \mathbb{Z}} \{ x \in P(B) : \delta(\mu)^T x \leq \delta_0(\mu) \}. \]
Proposition

\[ SC(B) = \bigcap_{\mu \in \mathcal{L}(B), \mu^T b \notin \mathbb{Z}} \{ x \in P(B) : \delta(\mu)^T x \leq \delta_0(\mu) \}. \]

Proposition

For \( \mu \in \mathcal{L}(B) \) s.t \( \mu^T b \notin \mathbb{Z} \) split cut

\[ (\mu^-)^T (Bx - b) + (1 - f(\mu^T b))(\mu^T Bx - \lfloor \mu^T b \rfloor) \leq 0 \]

dominates

\[ [\mu^-]^T (Bx - b) + (1 - f(\mu^T b))(\mu^T Bx - \lfloor \mu^T b \rfloor) \leq 0 \]
Studying $\mathcal{L}(B)$ in Each Orthant Decomposes $SC(B)$ to the Intersection of a *Finite* Number of Sets

For $\sigma \in \{0, 1\}^n$ let

$$\mathcal{L}(B, \sigma) := \{ \mu \in \mathcal{L}(B) : (-1)^{\sigma_i} \mu_i \geq 0, \quad \forall i \in \{1, \ldots, n\} \}$$

so that

$$SC(B) = \bigcap_{\sigma \in \{0, 1\}^n} SC(B, \sigma)$$

where

$$SC(B, \sigma) = \bigcap_{\mu \in \mathcal{L}(B, \sigma)} \{ x \in P(B) : \delta(\mu)^T x \leq \delta_0(\mu) \}$$
Studying $\mathcal{L}(B, \sigma)$ Allows Detecting Dominated Cuts

**Lemma**

Let $\sigma \in \{0, 1\}^n$ and let $\mu \in \mathcal{L}(B, \sigma)$ with $\mu = \alpha + \beta$ for $\alpha, \beta \in \mathcal{L}(B, \sigma)$ such that $\beta^T b \in \mathbb{Z}$. Then $\delta(\mu)^T x \leq \delta_0(\mu)$ is dominated by $\delta(\alpha)^T x \leq \delta_0(\alpha)$ in $P(B)$.

**Proof.**

Uses the fact that for $\alpha, \beta$ in the same orthant

$|\alpha + \beta| = |\alpha| + |\beta|$. 
A Finite Integral Generating Set (FIGS) of $\mathcal{L}(B, \sigma)$ Induces a Finite Subset of $\mathcal{L}(B, \sigma)$

- Let $\{v^i\}_{i \in \mathcal{V}(\sigma)} \subseteq \mathcal{L}(B, \sigma)$ be a (FIGS), i.e. a finite set such that

  \[ \mathcal{L}(B, \sigma) = \{ \mu \in \mathbb{R}^r : \mu = \sum_{i \in \mathcal{V}(\sigma)} k_i v^i, \; k_i \in \mathbb{Z}_+ \} \]

- We want $\mu^T b \notin \mathbb{Z}$, so for $i \in \mathcal{V}(\sigma)$ let

  \[ m_i = \min\{m \in \mathbb{Z}_+ \setminus \{0\} : mb^T v^i \in \mathbb{Z} \} \]

and define the following finite subset of $\mathcal{L}(B, \sigma)$.

\[ \mathcal{L}^0(B, \sigma) := \{ \mu \in \mathcal{L}(B, \sigma) : \mu = \sum_{i \in \mathcal{V}(\sigma)} r_i v^i, \; r_i \in \{0, \ldots, m_i-1\} \} \]
A Finite Integral Generating Set (FIGS) of $\mathcal{L}(B, \sigma)$ Induces a Finite Subset of $\mathcal{L}(B, \sigma)$

- Let $\{v^i\}_{i \in \mathcal{V}(\sigma)} \subseteq \mathcal{L}(B, \sigma)$ be a FIGS, i.e. a finite set such that

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- We want $\mu^T b \notin \mathbb{Z}$, so for $i \in \mathcal{V}(\sigma)$ let

$$m_i = \min\{m \in \mathbb{Z}_+ \setminus \{0\} : m b^T v^i \in \mathbb{Z}\}$$

and define the following finite subset of $\mathcal{L}(B, \sigma)$.

$$\mathcal{L}^0(B, \sigma) := \{\mu \in \mathcal{L}(B, \sigma) : \mu = \sum_{i \in \mathcal{V}(\sigma)} r_i v^i, \ r_i \in \{0, \ldots, m_i-1\}\}$$
Proving the Polyhedrality of $SC(B, \sigma)$ Yields the Polyhedrality of $SC$

**Theorem**

$SC(B, \sigma)$ the polyhedron given by

$$SC(B, \sigma) = \bigcap_{\mu \in \mathcal{L}^0(B, \sigma)} \{ x \in P(B) : \delta(\mu)^T x \leq \delta_0(\mu) \}$$

**Corollary**

$SC(B)$ is a polyhedron for all $B \in \mathcal{B}$. $SC$ is a polyhedron.
Proof Idea.

Goal: For \( \mu \in \mathcal{L}(B, \sigma) \), \( \delta(\mu)^T x \leq \delta_0(\mu) \) is dominated by \( \delta(\alpha)^T x \leq \delta_0(\alpha) \) for some \( \alpha \in \mathcal{L}^0(B, \sigma) \).

How:
- For \( \mu \in \mathcal{L}(B, \sigma) \) show that \( \mu = \alpha + \beta \) for \( \alpha, \beta \) such that:
  - \( \alpha \in \mathcal{L}^0(B, \sigma) \), \( \beta \in \mathcal{L}(B, \sigma) \)
  - \( \beta^T b \in \mathbb{Z} \)
- Use Lemma.
Proof Idea.

Goal: For $\mu \in \mathcal{L}(B, \sigma)$, $\delta(\mu)^T x \leq \delta_0(\mu)$ is dominated by $\delta(\alpha)^T x \leq \delta_0(\alpha)$ for some $\alpha \in \mathcal{L}^0(B, \sigma)$.

How:
- For $\mu \in \mathcal{L}(B, \sigma)$ show that $\mu = \alpha + \beta$ for $\alpha, \beta$ such that:
  - $\alpha \in \mathcal{L}^0(B, \sigma)$, $\beta \in \mathcal{L}(B, \sigma)$
  - $\beta^T b \in \mathbb{Z}$
- Use Lemma.
Proof of Theorem.

Let \( \{v^i\}_{i \in \mathcal{V}(\sigma)} \) be a FIGS for \( \mathcal{L}(B, \sigma) \) and let \( \{k_i\}_{i \in \mathcal{V}(\sigma)} \subseteq \mathbb{Z}_+ \) be such that

\[
\mu = \sum_{i \in \mathcal{V}(\sigma)} k_i v^i.
\]
Proof of Theorem.

Let \( \{v^i\}_{i \in \mathcal{V}(\sigma)} \) be a FIGS for \( \mathcal{L}(B, \sigma) \) and let \( \{k_i\}_{i \in \mathcal{V}(\sigma)} \subseteq \mathbb{Z}_+ \) be such that

\[
\mu = \sum_{i \in \mathcal{V}(\sigma)} k_i v^i.
\]

For each \( i \in \mathcal{V}(\sigma) \) we have

\[
k_i = n_i m_i + r_i
\]

for some \( n_i, r_i \in \mathbb{Z}_+, 0 \leq r_i < m_i \). Let

\[
\alpha = \sum_{i \in \mathcal{V}(\sigma)} r_i v^i \quad \text{and} \quad \beta = \sum_{i \in \mathcal{V}(\sigma)} n_i m_i v^i
\]

We have \( \alpha \in \mathcal{L}^0(B, \sigma) \) and, as \( m_i \) is such that \( m_i b^T v^i \in \mathbb{Z} \) we have \( b^T \beta \in \mathbb{Z} \).
Final Remarks

The proof of the Theorem gives a way of enumerating the inequalities of $SC(B, \sigma)$, $SC(B)$ and $SC$:
- Not practical for anything buy toy problems.
- There is redundancy in the enumeration for $SC$ and $SC(B)$.
- There is also redundancy in the enumeration of $SC(B, \sigma)$. In fact we can reduce $\mathcal{L}^0(B, \sigma)$ to

$$\mathcal{L}^0(B, \sigma) := \{ \mu \in \mathcal{L}(B, \sigma) : \mu = \sum_{i \in \mathcal{V}(\sigma)} r_i v^i, r_i \in \{0, \ldots, m_i-1\} \text{ and } \{r_i\}_{i \in \mathcal{V}(\sigma)} \text{ are relatively prime} \}$$

[Dash et. al. 2005] also give a constructive characterization with similar properties.
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$$L^0(B, \sigma) := \{ \mu \in L(B, \sigma) : \mu = \sum_{i \in V(\sigma)} r_i v^i, r_i \in \{0, \ldots, m_i-1\} \}$$

and $\{r_i\}_{i \in V(\sigma)}$ are relatively prime

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  - There is redundancy in the enumeration for $SC$ and $SC(B)$.
  - There is also redundancy in the enumeration of $SC(B, \sigma)$. In fact we can reduce $L^0(B, \sigma)$ to

$$L^0(B, \sigma) := \{ \mu \in L(B, \sigma) : \mu = \sum_{i \in V(\sigma)} r_i v^i, r_i \in \{0, \ldots, m_i - 1\} \text{ and } \{r_i\}_{i \in V(\sigma)} \text{ are relatively prime} \}$$

- [Dash et. al. 2005] also give a constructive characterization with similar properties.

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