6.046 Recitation 9 Handout

April 18, 2008

1 Convex Hull

1.1 The Problem

The convex hull of a set Q of points denoted $CH(Q)$ is the smallest convex polygon P for which each point in Q is either on the boundary of P or in its interior. See Figure 1 for an example.

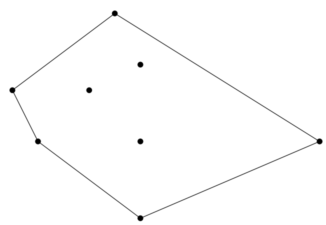

Figure 1: A set of points and its convex hull.

1.2 Graham's Scan

Sketch of the algorithm:

- Maintain a **stack** S of candidate points
- Each point in Q is pushed once onto the stack, and wrong points are eventually popped from the stack.
- At the end, stack S contains exactly the vertices of $CH(Q)$ in counterclockwise order.

Pseudocode:

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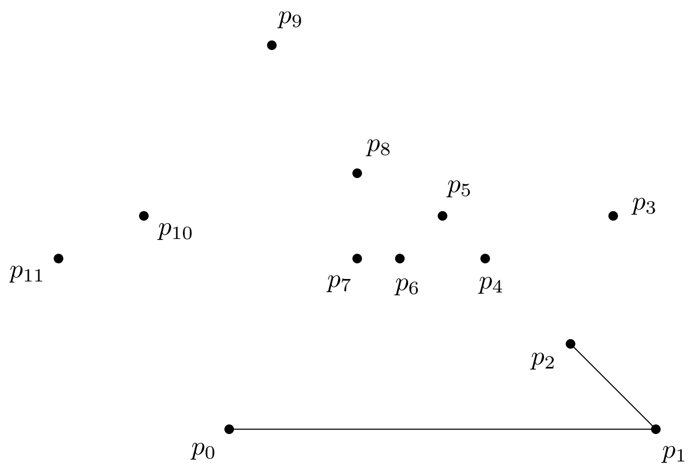
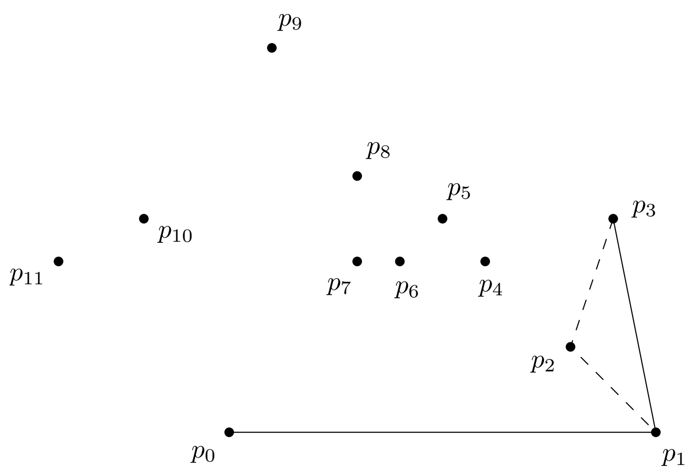
Graham-Scan(Q):

Let p_0 be the point in Q with the minimum y-coordinate or the
  leftmost point in case of a tie
Let {p_1, p_2, ... , p_m} be the remaining points in Q, sorted by
  polar angle in CCW order around p_0 (if more than one point has
  the same angle, remove all but the one that is farthest from p_0)
PUSH(p_0, S)
PUSH(p_1, S)
PUSH(p_2, S)
for i <- 3 to m
    do while the angle formed by points NEXT-TO-TOP(S), TOP(S), and
      p_i makes a non-left turn (straight or to the right)
        do POP(S)
    PUSH(p_i, S)
return S
```

Example of Graham's Scan:



At the beginning of the for loop, S contains p_0, p_1, and p_2.



When considering p_3, we first pop p_2 (because $\angle p_1 p_2 p_3$ makes a non-left turn) and then push p_3 onto the stack.

1.3 Analysis of Graham's Scan

1.3.1 Correctness

Theorem 1. *If* GRAHAM-SCAN *is run on a set Q of points, where $|Q| \geq 3$, then at termination, the stack S consists of, from bottom to top, exactly the vertices of $CH(Q)$ in counterclockwise order.*

Proof. We shall use the following loop invariant to prove this theorem:

> The stack S consists of, from bottom to top, exactly the vertices of $CH(Q_{i-1})$ in counterclockwise order, where Q_i is the set of points $\{p_0, p_1, \ldots, p_i\}$.

- **Initialization:** At the beginning of the loop, $i = 3$, and stack S consists of p_0, p_1, and p_2, which is exactly the contents of Q_{i-1}, or Q_2. The convex hull $CH(Q_2)$ is clearly the set of points in S because the convex hull of three points is going to be the triangle formed by the three points.

- **Maintenance:** Assuming that the loop invariant hold for all previous iterations of the loop, we show that on the i-th iteration of the loop, if the loop invariant is true at the beginning of the loop, it is therefore true at the beginning of the next one as well.

  Let p_j and p_k be the two points at the top of the stack right before p_i is pushed onto the stack.

  Claim 1. *After p_i is pushed onto the stack, S consists of $CH(Q_j \cup \{p_i\})$.*

  Proof. By the loop invariant, we know that S consists of $CH(Q_j)$ right before we push p_i onto the stack. This is because the stack is in the same state as directly after pushing p_j onto the stack. This is because no point is ever pushed more than once onto the stack, and thus no points beneath p_j could have been removed without popping p_j off the stack.

  Secondly, because we exited the while loop, we know that $\angle p_k p_j p_i$ is a left turn. Since p_0 is the bottom and leftmost point, we know that $\angle p_j p_i p_0$ is also a left turn. Thus, once we push p_i onto the stack, we get the convex hull of Q_j and the point p_i. □

  We still need to show that this convex hull is also the convex hull of Q_i, i.e. including the points between p_j and p_i that were popped off in the while loop.

  Claim 2. $CH(Q_j \cup \{p_i\}) = CH(Q_i)$.

  Proof. Let p_t be a point that was popped off and p_r be the point directly below it in the stack at that time. To have popped p_t, then $\angle p_r p_t p_i$ is a non-left turn. If it was a right turn, that would make the polygon not convex since we are going counterclockwise around the set of points, and p_t is not going to be on the perimeter of th econvex hull. If it was not a turn at all (the angle is 180°), then p_t is on the perimeter of the convex hull but is not going to be a vertex of the polygon. In either case, we can remove p_t from consideration and still have the same convex hull, i.e. $CH(Q_i - p_t) = CH(Q_i)$. This is true for all members of the set P_i of all points popped off during the i-th iteration of the loop, so we have $CH(Q_i - P_i) = CH(Q_i)$. Because we are using a stack in which all the points popped off had to be contiguous in the stack and we stopped popping when the top point in the stack was p_j, we have $CH(Q_i - P_i) = CH(Q_j \cup \{p_i\}) = CH(Q_i)$. □

- **Termination:** When the loop terminates, $i = m + 1$, so S consists of $CH(Q_m)$ by the loop invariant, which is equivalent to $CH(Q)$. This is because any points that were removed going from Q to Q_m in line 2 were not going to be in $CH(Q)$. (You should be able to see why this is pretty easily; if not, draw a picture.)

□

1.3.2 Runtime

- Line 1: $O(n)$
- Line 2: $O(n \lg n)$ for sorting
- Lines 3-5: $O(1)$
- Lines 6-9, calls to Push: $O(n)$ (each point only gets pushed once)
- Lines 6-9, calls to Pop: $O(n)$ (each point gets popped at most one time)
- Total running time: $O(n \lg n)$

1.4 Lower Bound

We get a lower bound of $\Omega(n \lg n)$ time by using the solution to a convex hull problem to sort numbers.

1. Given a list of numbers $(x_1, x_2, \ldots, x_n)$, construct the set with the points:$((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n))$ where $y_i = x_i^2$. This takes $O(n)$ time. The parabola is a convex function, which means that if all of our points are on the parabola (which they will be), they must all be part of the convex hull.
2. Find a convex hull on those points. This takes ??? time.
3. Walk along points on the hull and get them in sorted order in $O(n)$ time.
4. We must have taken $\Omega(n \lg n)$ to sort in the comparison model, so since everything else only took $O(n)$ time, finding the convex hull had to take at least $\Omega(n \lg n)$.