UNDECIDABILITY PROBLEMS

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* (1) Playing with PCP

(Adapted from John Martin, *Introduction to Languages and the Theory of Computation*, 20.3)

Say you are given the following five PCP dominos:

#1	#2	#3	#4	#5
ab	ba	b	abb	a
aba	abb	ab	b	bab

a) Which domino(s) *could* be used first in a PCP solution? Why?

b) Which domino(s) *could* be used last in a PCP solution? Why?

c) Find a PCP solution with these dominos.
(Don't spend too much time on this part if it's taking too long!)

** (2) Silly PCP, Tricks are for Kids!

(From Michael Sipser, *Introduction to the Theory of Computation*, 2^{nd} ed., Problems 5.17, 5.19.)

* a) In a variation of PCP, each domino the top string has the same length as the bottom string. Show that this variation of PCP is decidable.

** b) Prove that PCP is decidable over the unary[*] alphabet $\Sigma = \{a\}$.

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[*] Compare the word "unary" to "binary," and note that the root for one is "un" (e.g. "unit," "universal,") while the root for two is "bi" ("biweekly").

** (2) TMs can feel useless, too

(Adapted from Michael Sipser, *Introduction to the Theory of Computation*, 2nd ed., Problem 5.13.)

A <u>useless state</u> in a Turing machine is one that is never entered on any input string.

Consider the problem of determining whether a Turing machine has any useless states.

a) Formulate this problem as a language:

$USELESS_{TM}$ =

b) Fill in the steps of the following proof that $USELESS_{TM}$ is undecidable:

For contradiction, assume that ___________________________________ by TM R.
Construct a new TM S that uses R to decide A_{TM}.

S creates a new TM T that has a useless state when M doesn't accept w, and does not have a useless state when M does accept w.

S = "On input _____________________ :

1. Construct a new TM T = "On input x:
    a. Replace x on the input by the string <M, w>
    b. Run the universal TM U to simulate. (Note that U was designed to use all its states.)
    c. If U accepts, enter a special state q_A and <u>accept</u>.
2. Run R on ______________ to determine whether T has any useless states.
3. If R rejects, then M _________ (accepts/rejects) w, so S _________ (accepts/rejects).
    Otherwise, S _________ (accepts/rejects).

If M accepts w, then T enters all states, but if M doesn't accept w then T avoids q_A.
So T has a useless state, q_A, if and only if M doesn't accept w.

Thus S decides A_{TM}. Because A_{TM} is _________________, we have reached a _________________ and conclude that _________________.