Definition of a finite limit

A sequence \( \{a_n\} \) has a finite limit \( L \) if for all \( \varepsilon > 0 \), there exists a natural number \( N \) (which is a function of \( \varepsilon \)), such that for all \( n > N \),

\[ |a_n - L| < \varepsilon \]

In math-speak:

A sequence \( \{a_n\} \) has a finite limit \( L \) if for all \( \varepsilon > 0 \), there exists a natural number \( N(\varepsilon) \) such that for all \( n > N(\varepsilon) \),

\[ |a_n - L| < \varepsilon \]

Inside... full proofs from your first calc tutorial!
Q1: From the definition of a limit, prove:

\[
\lim_{n \to \infty} \frac{n-7}{3n+8} = \frac{1}{3}
\]

A1: Let \( \varepsilon \) be any positive number.

We desire: \( |a_n - L| < \varepsilon \)

This happens iff: \( \left| \frac{n-7}{3n+8} - \frac{1}{3} \right| < \varepsilon \)

\[
\left| \frac{3(n-7) - 1 \cdot (3n+8)}{3(3n+8)} \right| < \varepsilon 
\]

\[
\left| \frac{3n - 21 - 3n - 8}{3(3n+8)} \right| < \varepsilon 
\]

\[
\left| \frac{-29}{3(3n+8)} \right| < \varepsilon 
\]

Note denominator always positive

\[
\frac{29}{3(3n+8)} < \varepsilon 
\]

\[
29 < \varepsilon \cdot 3(3n+8) 
\]

\[
29 < 9\varepsilon n + 24\varepsilon 
\]

\[
(29 - 24\varepsilon) / 9\varepsilon < n 
\]

In other words,

\[
|a_n - L| < \varepsilon \iff n > (29 - 24\varepsilon) / 9\varepsilon 
\]
A7 (continued) \[ \text{Now, in order for } L \text{ to qualify as a finite limit, we must show that there exists some natural number } N \text{ (which is a function of } \varepsilon) \text{ such that for any } n > N, \quad |a_n - L| < \varepsilon \]

Does such an \( N \) exist? \textbf{Yes!} Just pick \( N \) to be any natural number greater than \( \frac{(29 - 24\varepsilon)}{9\varepsilon} \)

Then, \( n > N \Rightarrow n > \frac{(29 - 24\varepsilon)}{9\varepsilon} \)

... which we just shows guarantees that \( |a_n - L| < \varepsilon \), as desired:

\textbf{Summary:} For any } \varepsilon > 0 \text{ we have shown there exists a natural number } N \text{ (which is a function of } \varepsilon) \text{ such that for all } n > N, \quad |a_n - L| < \varepsilon \]

Thus, by definition, we have shown that \( L \) is the finite limit of the sequence.

\[ a_n = \frac{n - 3}{3n + 8} \]

\[ L = \frac{1}{3} \]

\[ \varepsilon \]

\[ |a_n - L| < \varepsilon \text{ for all } n > N(\varepsilon), \quad |a_n - L| < \varepsilon \]

That means "QED" or "OK, I'm done with the proof now, thanks!"
Q2: From the definition of a finite limit, prove:

$$\lim_{n \to \infty} \frac{2n+5}{n+1} \neq 1$$

A2: Let's try to "prove" $L = 1$ is a limit for $\frac{2n+5}{n+1}$ (We'll find that it's impossible...)

Let $\epsilon$ be any positive number.

We desire: $|a_n - L| < \epsilon$

This happens iff: $\left| \frac{2n+5}{n+1} - 1 \right| < \epsilon$

$$\left| \frac{2n+5 - n - 1}{n+1} \right| < \epsilon$$

$$\left| \frac{2n+5 - n-1}{n+1} \right| < \epsilon$$

$$\left| \frac{n+4}{n+1} \right| < \epsilon$$

Note both numerator & denominator are always positive

$$\frac{n+4}{n+1} < \epsilon$$

$n+4 < \epsilon(n+1)$

$n+4 < 3n+\epsilon$

$n-\epsilon n < \epsilon-4$

$n(1-\epsilon) < \epsilon-4$
A2: Now we split into 3 cases:

1. $0 < \varepsilon < 1$
2. $\varepsilon > 1$
3. $\varepsilon = 1$

Note by construction, $\varepsilon > 0$

We split into cases since the signs of $(1-\varepsilon)$ and $(\varepsilon-1)$ will affect the direction of the inequality.

Case 1: $0 < \varepsilon < 1$:

\[
\frac{n(1-\varepsilon)}{\varepsilon - 1} < n < \frac{\varepsilon - 1}{1-\varepsilon}
\]

positive! positive!

How did we get here? We started by saying we want $|a_n - L| < \varepsilon$, and that brought us to the requirement that $n < \frac{\varepsilon - 1}{1-\varepsilon}$.

In other words, $|a_n - L| < \varepsilon \iff n < \frac{\varepsilon - 1}{1-\varepsilon}$

Now, in order for $L = 1$ to qualify as a finite limit, we must show that there exists some natural number $N$ (which is a function of $\varepsilon$) such that for every $n > N$, $|a_n - L| < \varepsilon$.

But we just showed that $|a_n - L| < \varepsilon$ iff $n < \frac{\varepsilon - 1}{1-\varepsilon}$.

So there's NO WAY there exists the desired $N$.

Thus, by definition, $L$ cannot be a finite limit for the sequence.
A2:  But what about the other two cases for $\varepsilon$'s value — $\varepsilon > 1$ and $\varepsilon = 1$?

**WE DON'T CARE** Sure, we might (or might not) be able to find suitable $N$'s for these other values of $\varepsilon$ — showing that for these values of $\varepsilon$, $L$ is kind of like a limit.

**BUT... the limit definition** says quite simply that for all $\varepsilon > 0$ there exists a natural number $N$ such that ... blah blah blah.

We showed that $L$ is not the limit for $0 < \varepsilon < 1$, so $L$ is not the limit for all $\varepsilon > 0$.

So $L$ is just not the limit! (For this sequence.)

In case you're curious, the limit is $2.0$.

\[ L^n \]

\[ an, a_{n-1}, a_{n-2}, \ldots \leq |an-1| \geq \varepsilon \text{ always} \]

So if $0 < \varepsilon < 1$,

\[ |an-1| \geq \varepsilon \text{ always} \]

Never find an $N$ for which all $n > N$ give $|an-1| < \varepsilon$.