Definition of a finite limit

> A sequence $\{a_n\}$ has a finite limit L if for all $\varepsilon > 0$, there exists a natural number N (which is a function of ε), such that for all $n > N$,
> $$|a_n - L| < \varepsilon$$

In math-speak:

> A sequence $\{a_n\}$ has a finite limit L if $\forall \varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ s.t. $\forall n > N$,
> $$|a_n - L| < \varepsilon$$

Inside... full proofs from your first calc tutorial!

Q1: From the definition of a limit, prove:

$$\lim_{n\to\infty} \frac{n-7}{3n+8} = \frac{1}{3}$$

A1: Let ε be <u>any</u> positive number.

We desire: $|a_n - L| < \varepsilon$

This happens iff: $\left|\frac{n-7}{3n+8} - \frac{1}{3}\right| < \varepsilon$

$$\left|\frac{3(n-7) - 1\cdot(3n+8)}{3(3n+8)}\right| < \varepsilon$$

$$\left|\frac{3n-21-3n-8}{3(3n+8)}\right| < \varepsilon$$

$$\left|\frac{-29}{3(3n+8)}\right| < \varepsilon$$

Note denominator always positive

$$\frac{29}{3(3n+8)} < \varepsilon$$

$$29 < \varepsilon \cdot 3(3n+8)$$

$$29 < 9\varepsilon n + 24\varepsilon$$

$$(29-24\varepsilon)/9\varepsilon < n$$

In other words,

$$|a_n - L| < \varepsilon \iff n > (29-24\varepsilon)/9\varepsilon$$

A1: continued

Now, in order for L to qualify as a finite limit, we must show that there exists some natural number N (which is a function of ε) such that for any $n > N$, $|a_n - L| < \varepsilon$

Does such an N exist? Yes! Just pick N to be any natural number greater than $(29 - 24\varepsilon)/9\varepsilon$

Then, $n > N \Rightarrow n > (29 - 24\varepsilon)/9\varepsilon$

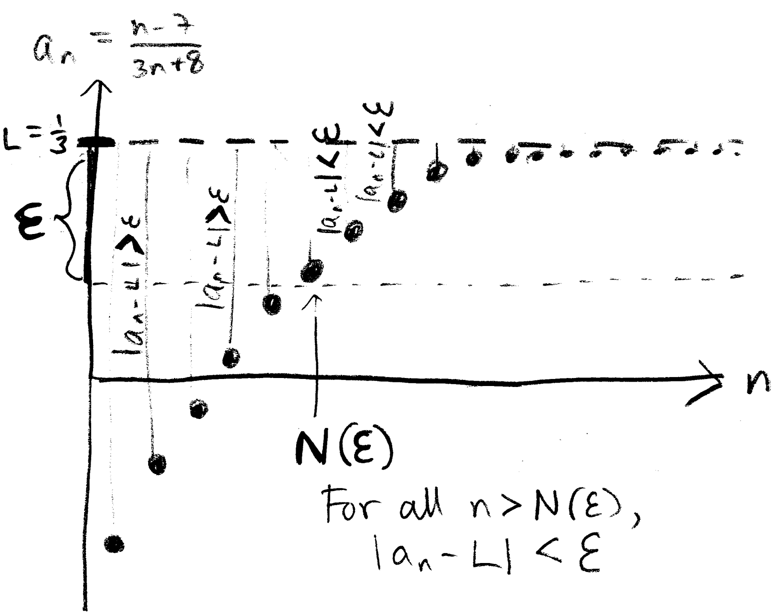
... which we just shows guarantees that $|a_n - L| < \varepsilon$, as desired.

Summary: For any $\varepsilon > 0$ we have shown there exists a natural number N (which is a function of ε) such that for all $n > N$,

$$|a_n - L| < \varepsilon$$

We can say "for any $\varepsilon > 0$" since we never restricted the positive values that ε can be.

Thus, by definition, we have shown that L is the finite limit of the sequence.



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That means "QED" or "OK, I'm done with the proof now, thanks!"

Q2: From the definition of a finite limit, prove:

$$\lim_{n\to\infty} \frac{2n+5}{n+1} \neq 1$$

A2: Let's try to "prove" $L=1$ is a limit for $\left\{\frac{2n+5}{n+1}\right\}$
(We'll find that it's impossible...)

Let ε be any positive number.

We desire: $|a_n - L| < \varepsilon$

This happens iff: $\left|\frac{2n+5}{n+1} - 1\right| < \varepsilon$

$$\left|\frac{2n+5-1\cdot(n+1)}{n+1}\right| < \varepsilon$$

$$\left|\frac{2n+5-n-1}{n+1}\right| < \varepsilon$$

$$\left|\frac{n+4}{n+1}\right| < \varepsilon$$

Note both numerator & denominator are always positive

$$\frac{n+4}{n+1} < \varepsilon$$

$$n+4 < \varepsilon(n+1)$$

$$n+4 < \varepsilon n + \varepsilon$$

$$n - \varepsilon n < \varepsilon - 4$$

$$n(1-\varepsilon) < \varepsilon - 4$$

A2: continued

Now we split into 3 cases:

1. $0 < \varepsilon < 1$
2. $\varepsilon > 1$
3. $\varepsilon = 1$

Note by construction, $\varepsilon > 0$

We split into cases since the signs of $(1-\varepsilon)$ and $(\varepsilon - 4)$ will affect the direction of the inequality

Case 1: $0 < \varepsilon < 1$:

$$\underbrace{n(1-\varepsilon)}_{\text{positive!}} < \underbrace{\varepsilon - 4}_{\text{positive!}}$$

$$n < \frac{\varepsilon - 4}{1-\varepsilon}$$

How did we get here? We started by saying we want $|a_n - L| < \varepsilon$, and that brought us to the requirement that $n < \frac{\varepsilon - 4}{1-\varepsilon}$

In other words, $|a_n - L| < \varepsilon \iff n < \frac{\varepsilon - 4}{1-\varepsilon}$

Now, in order for $L = 1$ to qualify as a finite limit, we must show that there exists some natural number N (which is a function of ε) such that for every $n > N$, $|a_n - L| < \varepsilon$

But we just showed than $|a_n - L| < \varepsilon$ iff $n < \frac{\varepsilon - 4}{1-\varepsilon}$
So there's NO WAY there exists the desired N.
Thus, by definition, L cannot be a finite limit for the sequence.

A2: (continued) But what about the other two cases for ε's value — $\varepsilon > 1$ and $\varepsilon = 1$?

WE DON'T CARE. Sure, we might (or might not) be able to find suitable N's for these other values of ε — showing that for <u>these values of ε</u>, L is kind of like a limit.

BUT... the limit definition says quite plainly that <u>for all $\varepsilon > 0$</u> there exists a natural number N such that ... blah blah blah.
N must work for <u>all</u> positive values of ε, not just <u>some</u>.

We showed that L is <u>not</u> the limit for $0 < \varepsilon < 1$, so L is not the limit for <u>all</u> $\varepsilon > 0$.
So L is just not the limit! (For this sequence.)

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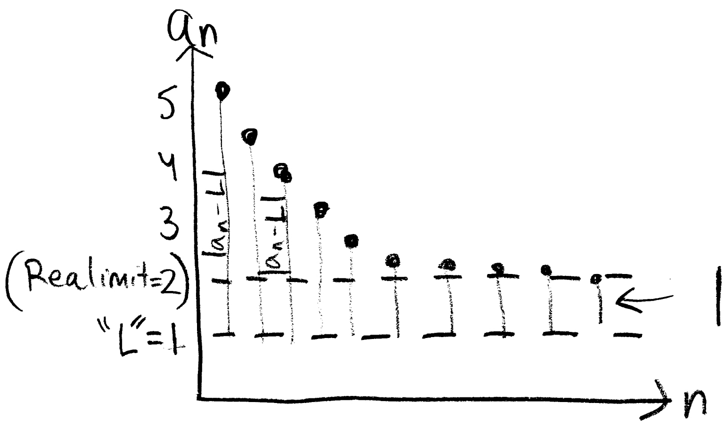
In case you're curious, the limit is 2. ◡̈



$\leftarrow |a_n - 1| > 1$ <u>always</u>

So if $0 < \varepsilon < 1$,
$|a_n - 1| > \varepsilon$ always

Never find an N for which all $n > N$ give $|a_n - 1| < \varepsilon$