Even More Limits, and Continuity

104003 Differential and Integral Calculus I
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Algebra of Limits:

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = K$, then:

- **Sum:** $\lim_{x\to a}\big(f(x) + g(x)\big) = L + K$
- **Product:** $\lim_{x\to a}\big(f(x) * g(x)\big) = L * K$
- **Quotient:** $\lim_{x\to a}(f(x)/g(x)) = L/K$ (if g and K are both non-0)
- **Multiplication by a constant:** $\lim_{x\to a}\big(c * f(x)\big) = c * L$ (where c is any constant)

One-sided Limits (Definition)

From the left/below:
$\lim_{x\to a-} f(x) = L$ iff: for all ε > 0, there exists a δ >0 such that for all $x \neq a$,

$$0 < a - x < \delta \quad \Leftrightarrow \quad |\, f(x) - L \,| < \varepsilon$$

From the right/above:
$\lim_{x\to a+} f(x) = L$ iff: for all ε > 0, there exists a δ >0 such that for all $x \neq a$,

$$0 < x - a < \delta \quad \Leftrightarrow \quad |\, f(x) - L \,| < \varepsilon$$

(Recall definition for a limit was similar to the above, just had $0 < |x - a| < \delta$ … note the absolute value.)

Thm: $f(x)$ has a finite limit at $x = a$ iff both 1-sided limits exist *and* are equal to each other.

Finite Limit as x→∞ (Definition)

$\lim_{x\to\infty} f(x) = L$ iff: for all ε > 0, there exists x_0 such that for all $x > x_0$,

$$|\, f(x) - L \,| < \varepsilon$$

"Infinite" Limit as x→a (Definition)

$\lim_{x\to a} f(x) = \infty$ iff: for all M > 0, there exists a δ >0 such that for all x,

$$0 < |x - a| < \delta \quad \Rightarrow \quad f(x) > M$$

"Infinite" Limit as x→∞ (Definition)

$\lim_{x\to\infty} f(x) = \infty$ iff: for all M > 0, there exists x_0 such that for all $x > x_0$,

$$f(x) > M$$

- Similar definitions for lim = - ∞ , just M < 0 and $f(x) < M$.
- Note "infinite" limits do *not* exist. We just write lim = ±∞ to easily say why the limit doesn't exist (because f(x) increases/decreases without bound).

<u>Continuity</u> (Definition)

Let a be a point in the domain of function $f(x)$.

Then ***f* is continuous at *x = a*** iff:

1) $f(a)$ is defined
2) $\lim_{x \to a} f(x)$ exists (i.e. is finite)
3) $\lim_{x \to a} f(x) = f(a)$

(Informally, the <u>limit</u> of f at a equals the <u>value</u> of f at a.)

Alternative (but equivalent) definition of continuity:

f is **continuous at *x = a*** iff for every ε > 0 there exists δ > 0 such that for all x:

$$|x - a| < \delta \iff |f(x) - f(a)| < \varepsilon$$

Function ***f* is continous** iff it is continous at every point of its domain.

<u>Algebra of Continuity</u>

If f(x) and g(x) are both continuous at x=a, then the following are also continous at x=a:

- **Sum:** f(x) + g(x)
- **Product:** f(x) * g(x)
- **Quotient:** f(x) / g(x) (if g(x=a) ≠ 0)
- **Composition:** f(g(x)) = (f○g)(x) (where c is any constant)

<u>Continuous Functions</u>

The elementary functions are continuous.

Forgot what elementary functions are? Recall:

The 5 Elementary Operations: $+$, $-$, $\times$, $\div$, composition [composition: f (g(x)) = (f ○ g)(x)]

Elementary Functions: The functions we get from:

c *(const)* x a^x sin(x) arcsin(x) $\log_a(x)$

… and their combinations through the elementary operations

Thus, **polynomial functions and rational functions (when denominator ≠ 0) functions are always continuous.**