Algebra of Limits:

If \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = K \), then:

- **Sum:** \( \lim_{x \to a} (f(x) + g(x)) = L + K \)
- **Product:** \( \lim_{x \to a} (f(x) \cdot g(x)) = L \cdot K \)
- **Quotient:** \( \lim_{x \to a} \left( \frac{f(x)}{g(x)} \right) = \frac{L}{K} \) (if \( g \) and \( K \) are both non-0)
- **Multiplication by a constant:** \( \lim_{x \to a} (c \cdot f(x)) = c \cdot L \) (where \( c \) is any constant)

One-sided Limits (Definition)

**From the left/below:**
\[
\lim_{x \to a^-} f(x) = L \quad \text{iff:} \quad \text{for all } \varepsilon > 0, \text{ there exists a } \delta > 0 \text{ such that for all } x \neq a,
0 < a - x < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon
\]

**From the right/above:**
\[
\lim_{x \to a^+} f(x) = L \quad \text{iff:} \quad \text{for all } \varepsilon > 0, \text{ there exists a } \delta > 0 \text{ such that for all } x \neq a,
0 < x - a < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon
\]

(Recall definition for a limit was similar to the above, just had \( 0 < |x - a| < \delta \) ... note the absolute value.)

**Thm:** \( f(x) \) has a finite limit at \( x = a \) iff both 1-sided limits exist and are equal to each other.

Finite Limit as \( x \to \infty \) (Definition)

\[
\lim_{x \to \infty} f(x) = L \quad \text{iff:} \quad \text{for all } \varepsilon > 0, \text{ there exists } x_0 \text{ such that for all } x > x_0,
|f(x) - L| < \varepsilon
\]

“Infinite” Limit as \( x \to a \) (Definition)

\[
\lim_{x \to a} f(x) = \infty \quad \text{iff:} \quad \text{for all } M > 0, \text{ there exists a } \delta > 0 \text{ such that for all } x,
0 < |x - a| < \delta \quad \Rightarrow \quad f(x) > M
\]

“Infinite” Limit as \( x \to \infty \) (Definition)

\[
\lim_{x \to \infty} f(x) = \infty \quad \text{iff:} \quad \text{for all } M > 0, \text{ there exists } x_0 \text{ such that for all } x > x_0,
f(x) > M
\]

- Similar definitions for \( \lim = -\infty \), just \( M < 0 \) and \( f(x) < M \).
- Note “infinite” limits do not exist. We just write \( \lim = \pm \infty \) to easily say why the limit doesn’t exist (because \( f(x) \) increases/decreases without bound).
**Continuity (Definition)**

Let $a$ be a point in the domain of function $f(x)$.

Then $f$ is **continuous at $x = a$** iff:

1) $f(a)$ is defined
2) $\lim_{x \to a} f(x)$ exists (i.e. is finite)
3) $\lim_{x \to a} f(x) = f(a)$

(Informally, the limit of $f$ at $a$ equals the value of $f$ at $a$.)

**Alternative (but equivalent) definition of continuity:**

$f$ is **continuous at $x = a$** iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x$:

$$|x - a| < \delta \iff |f(x) - f(a)| < \varepsilon$$

Function $f$ is **continuous** iff it is continuous at every point of its domain.

**Algebra of Continuity**

If $f(x)$ and $g(x)$ are both continuous at $x=a$, then the following are also continuous at $x=a$:

- **Sum**: $f(x) + g(x)$
- **Product**: $f(x) \times g(x)$
- **Quotient**: $f(x) / g(x)$     (if $g(x=a) \neq 0$)
- **Composition**: $f(g(x)) = (f \circ g)(x)$     (where $c$ is any constant)

**Continuous Functions**

The elementary functions are continuous.

Forgot what elementary functions are? Recall:

**The 5 Elementary Operations**: $+, -, \times, \div, \text{composition}$     \[\text{composition: } f(\ g(x)\ ) = (f \circ g)(x)\ ]

**Elementary Functions**: The functions we get from:

$$c (\text{const}) \quad x \quad a^x \quad \sin(x) \quad \arcsin(x) \quad \log_a(x)$$

... and their combinations through the elementary operations

Thus, **polynomial functions and rational functions (when denominator $\neq 0$) functions are always continuous.**