Taylor Polynomial & Max-Min Problems

104003 Differential and Integral Calculus I
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Tutorial Summary – February 16, 2011 – Kayla Jacobs

Taylor Polynomial

If function *f(x)* can be differentiated (at least) *n* times in the neighborhood of point *x* = *a*,
then the **n^th^-degree Taylor polynomial of *f*(x) at *x* = *a*** is:

$$T_n(x,a) = f(a) + \frac{f'(a)}{1!}\cdot(x-a) + \frac{f''(a)}{2!}\cdot(x-a)^2 + \frac{f^{(3)}(a)}{3!}\cdot(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}\cdot(x-a)^n$$

$$= \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}\cdot(x-a)^k$$

This is the **best possible *n*-degree approximation** of *f*(*x*) "near" *x* = *a*.
The more terms you include (the higher *n* is), and/or the closer to *x* = *a*… the better the approximation.

As $n \to \infty$, the Taylor polynomial **coverges to the exact function *f***.
It's then is called the "**Taylor series** of *f*." (Note then it doesn't matter how "near" you are to *x* = *a*.)

For $n = 1$, you simply get the **linear approximation** we've already learned about!
$\boldsymbol{f(x) \approx f(a) + f'(a)\cdot(x-a)}$ when $x \approx a$

Remainder: $R_n(x,a)$ is the **n^th^-degree remainder for f(x) at *x* = *a***.
This is the error made by the approximation of *f* as a Taylor polynomial:

$$R_n(x,a) = f(x) - T_n(x,a)$$

$$= \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$ … where *z* is a number between *a* and *x*.

Maclaurin Polynomial

Special case of Taylor polynomial, when ***a* = 0.** $T_n(x,0) = \sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}\cdot x^k$

Common Maclaurin series (as $n \to \infty$):

- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k$ (for $|x| < 1$)
- $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{k=0}^{\infty}(-1)^k\cdot x^k$ (for $|x| < 1$)
- $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\cdot x^{2k+1}$ (for all real x)
- $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\cdot x^{2k}$ (for all real x)
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty}\frac{x^k}{k!}$ (for all real x)

DeMoivre's Theorem

$e^{ix} = \cos(x) + i\cdot\sin(x)$ (derivable from the Maclaurin series for sin(x), cos(x), and e^x)

Maclaurin Polynomial Example: sin(x)

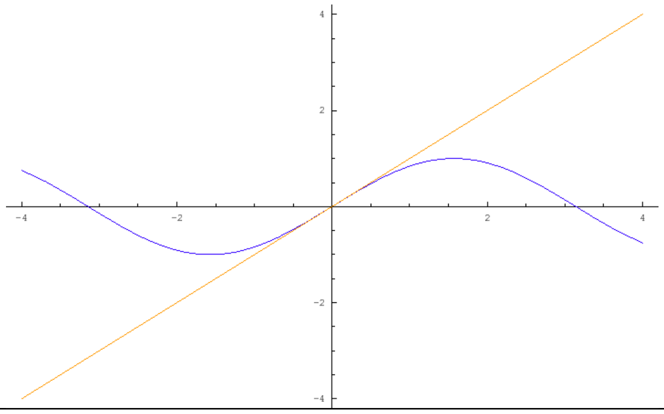
The graphs below show the actual function, $\sin(x)$, and its Maclaurin polynomial for various values of *n*.

As $n \to \infty$, the Maclaurin series is: $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot x^{2k+1}$

Note that there are no even-n terms, so $T_{2n}(x) = T_{2n-1}(x)$

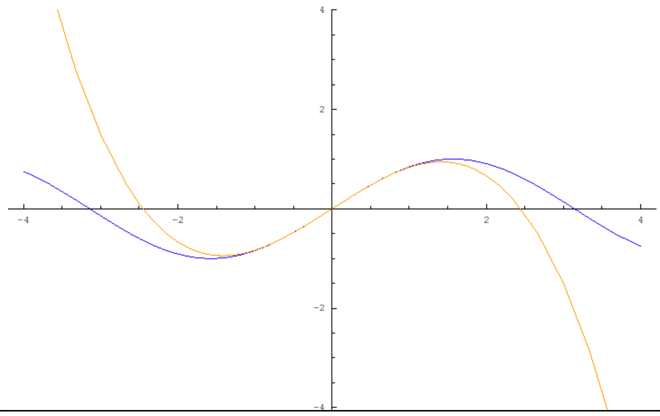
n = 1

$T_1(x) = x$



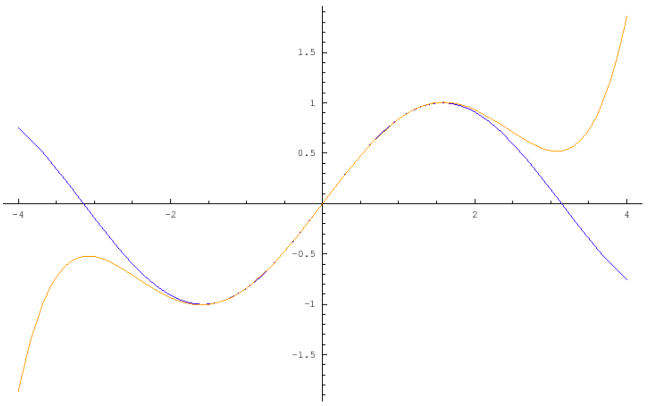
n = 3

$T_3(x) = x - \frac{x^3}{3!}$



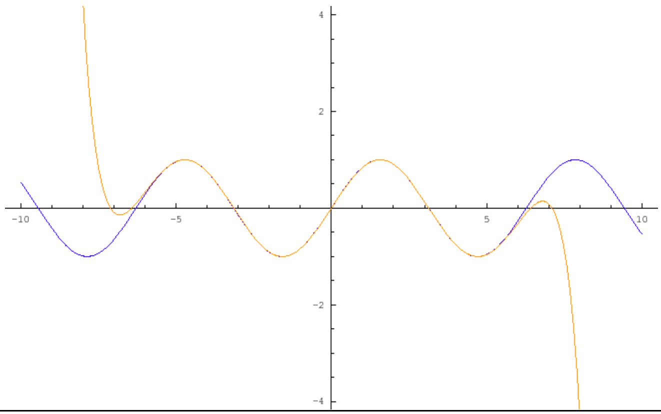
n = 5

$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$



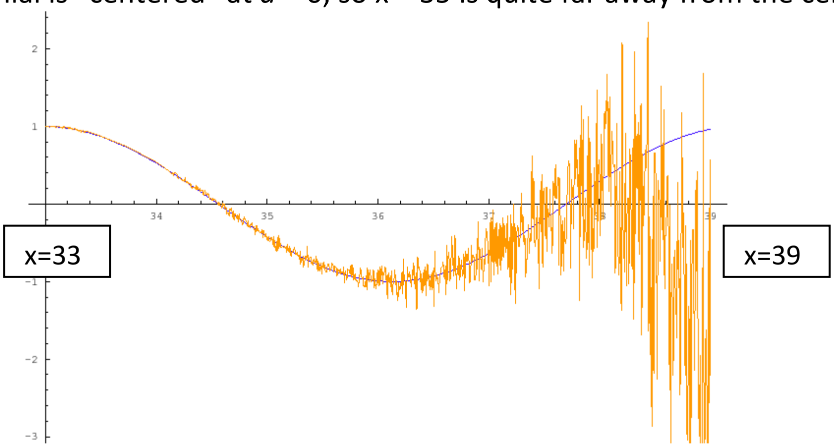
… n = 15

$T_{15}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!}$



Now take a look at **n = 99**. It agrees very well with the actual $\sin(x)$ curve until you get out to about x = 35, when it goes crazy. Here is a picture of the x-interval (34, 39):

(Remember, the polynomial is “centered” at *a* = 0, so x = 35 is quite far away from the center.)



Max-Min Problems

(Adapted from http://www.math.ucdavis.edu/~kouba/CalcOneDIRECTORY/maxmindirectory/MaxMin.html)

1. **Read each problem slowly and carefully.** If you misread the problem or hurry through it, you have NO chance of solving it correctly.

2. If appropriate, **draw a sketch** or diagram of the problem to be solved. Pictures are a great help in organizing and sorting out your thoughts.

3. **Define variables** to be used and carefully label your picture or diagram with these variables. This step is very important because it leads directly or indirectly to the creation of mathematical equations.

4. **Write down all equations** which are related to your problem or diagram. Clearly denote that equation which you are asked to maximize or minimize.

- Experience will show you that MOST optimization problems will begin with two equations. One equation is a **"constraint" equation** and the other is the **"optimization" equation**.

- **The "constraint" equation is used to solve for one of the variables. This is then substituted into the "optimization" equation before differentiation occurs.**

- Some problems may have NO constraint equation. Some problems may have two or more constraint equations.

5. Before differentiating, make sure that the **optimization equation is a function of only one variable**.

6. **Differentiate** using the well-known rules of differentiation. **Solve** for the variable value(s) that satisfy the **derivative being set to 0**.

6. **Verify that your result is a maximum or minimum** value using the first or second derivative test for extrema.

7. If appropriate, don't forget to **check the endpoints**, which might be the global maximum/minimum.