Convexity

Function $f$ is **convex** on interval $(a,b)$ if:

**Definition 1:**
At every point $x_1$ in $(a,b)$, $f$ lies **above the tangent line** at the point:

$$f(x) \geq f(x_1) + f'(x_1) \cdot (x - x_1)$$

**Definition 2:**
Between every 2 points $x_0$ and $x_1$ in $(a, b)$, $f$ lies **below the secant line** connecting the points:

$$f(t \cdot x_1 + (1 - t) \cdot x_2) \geq t \cdot f(x_1) + (1 - t) \cdot f(x_2) \quad \text{for } t \in [0,1]$$

- If $f''(x) \geq 0$ always on $(a,b)$, then $f$ is **convex** $\cup$ (according to both definitions above).
- If $f''(x) \leq 0$ always on $(a,b)$, then $f$ is **concave** $\cap$.
- If $f''(x) = 0$ always on $(a,b)$, then $f$ is a straight line, and is both concave and convex.

**Other notation:** concave = concave down $\cap$; convex = concave up $\cup$

**Inflection point:** When a function changes from being convex to concave, or vice versa. At an inflection point $x = a$, $f''(a)$ is either $0$ or doesn’t exist; and $f''(x)$ is negative (convex) on 1 side of $a$ and positive (concave) on the other side. An inflection point where $f''(a)$ doesn’t exist is a **corner point**.
Infinite Integral

- The **infinite integral** of function \( f \) is the **antiderivative**: the function \( F \) that, when differentiated, gives you \( f \). It's the reverse process of differentiation. \( \int f(x) \, dx = F(x) + c \iff F'(x) = f(x) \)

- Don’t forget the **\( +C \)** constant!

- If \( \int f \, dx = F_1 \) and \( \int f \, dx = F_2 \), then \( F_1 \) and \( F_2 \) are equal or differ by a constant \( (F_1 = F_2 + c) \).

- The integral is a **linear operator**: For any functions \( f \) and \( g \) and for any constant \( c \),

\[
\int (f + g) \, dx = \int f \, dx + \int g \, dx \\
\int (c \cdot f) \, dx = c \cdot \int f \, dx
\]

Integration by Parts

\[
\int f(x) \cdot g'(x) \, dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x) \, dx
\]

**Strategy**: Try to choose \( f \) and \( g' \) such that:

- \( f \) is simpler when differentiated
- \( g' \) is simpler when integrated
- \( \int f'(x) \cdot g(x) \, dx \) is simpler to integrate than \( \int f(x) \cdot g'(x) \, dx \)

Another way to write it: \( u = f(x) \) \quad \rightarrow \quad \frac{du}{dx} = f'(x) \quad \rightarrow \quad du = f'(x) \, dx \)

\( v = g(x) \) \quad \rightarrow \quad \frac{dv}{dx} = g'(x) \quad \rightarrow \quad dv = g'(x) \, dx \)

\[
\int u \, dv = uv - \int v \, du
\]

Integration of Rational Functions \( \int \frac{p(x)}{q(x)} \, dx \) Using Partial Fractions

1. Check that \( \text{degree}(p) < \text{degree}(q) \).
   - If it’s not, use polynomial division to rewrite \( p/q \) as a polynomial plus a new rational polynomial \( p_2/q_2 \) where \( \text{degree}(p_2) < \text{degree}(q_2) \).

Example:

\[
\frac{x^3 - 4x - 10}{x^2 - x - 6} = x + 1 + \frac{8x - 4}{x^2 - x - 6}.
\]

2. Factor the denominator \( q \) until you can’t anymore (factor into “irreducible terms”).

3. Write \( p/q \) as a sum of partial fractions (and solve for constants).

Examples:

\[
\frac{x - 1}{x^2 + x} = \frac{x - 1}{x(x + 1)} = \frac{A}{x} + \frac{B}{x + 1} \\
\frac{2x - 3}{x^3 + x} = \frac{2x - 3}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} \\
\frac{2x^3 + 5x - 1}{(x + 1)^2(x^2 + 1)^2} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{x^2 + 1} + \frac{Dx + E}{(x + 1)^3} + \frac{Fx + G}{(x^2 + 1)^2}
\]

4. Integrate! (It should be easier now.)