Convexity & Indefinite Integrals

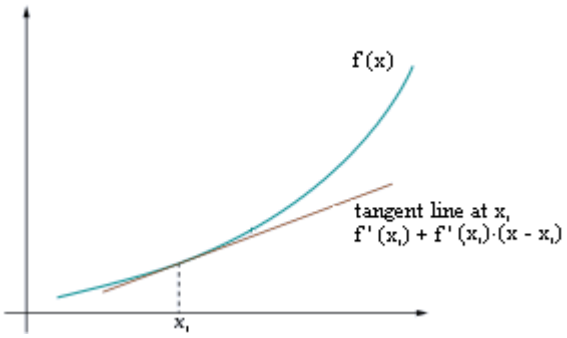
104003 Differential and Integral Calculus I
Technion International School of Engineering 2010-11
Tutorial Summary – February 20, 2011 – Kayla Jacobs

Convexity

Function *f* is **convex** on interval (a,b) if:

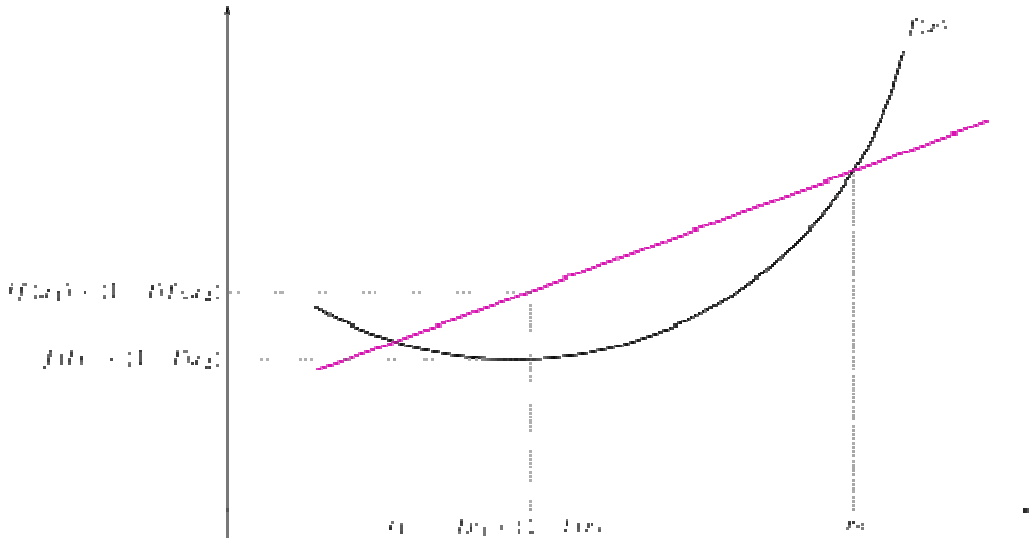
Definition 1:
At every point x_1 in (a,b), *f* lies **above the tangent line** at the point:

$$f(x) \geq f(x_1) + f'(x_1) \cdot (x - x_1)$$

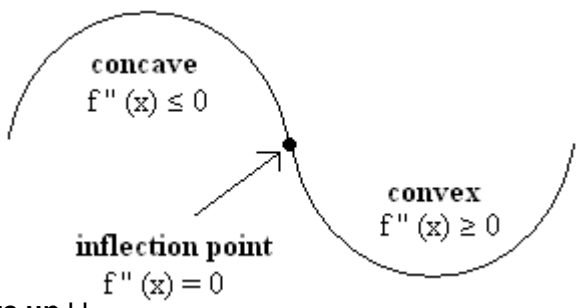


Definition 2:
Between every 2 points x_0 and x_1 in (a, b), *f* lies **below the secant line** connecting the points:

$$\boldsymbol{f}(t \cdot x_1 + (1-t) \cdot x_2) \geq \boldsymbol{t} \cdot \boldsymbol{f}(x_1) + (\mathbf{1} - \boldsymbol{t}) \cdot \boldsymbol{f}(x_2) \qquad \text{for } t \epsilon [0,1]$$



- If $f''(x) \geq 0$ always on (a,b), then *f* is **convex** ∪ (according to **both** definitions above).
- If $f''(x) \leq 0$ always on (a,b), then *f* is **concave** ∩.
- If $f''(x) = 0$ always on (a,b), then f is a straight line, and is **both concave and convex**.



Other notation: concave = **concave down** ∩; convex = **concave up** ∪

Inflection point: When a function changes from being convex to concave, or vice versa.
At an inflection point *x* = *a*, f''(a) is either 0 or doesn't exist;
and f ''(x) is negative (convex) on 1 side of a and positive (concave) on the other side.
An inflection point where f''(a) doesn't exist is a **corner point**.

Indefinite Integral

- The **indefinite integral** of function *f* is the **antiderivative:** the function *F* that, when differentiated, gives you *f*. It's the reverse process of differentiation. $\int f(x)\,dx = F(x) + c \quad \leftrightarrow \quad F'(x) = f(x)$
- Don't forget the **+C** constant!
- If $\int f\,dx = F_1$ <u>and</u> $\int f\,dx = F_2$, then F_1 and F_2 are equal or differ by a constant ($F_1 = F_2 + c$).
- The integral is a **<u>linear operator</u>:** For any functions *f* and *g* and for any constant *c*,

$$\int (f+g)\,dx = \int f\,dx + \int g\,dx$$

$$\int (c \cdot f)\,dx = c \cdot \int f\,dx$$

Integration by Parts

$$\boxed{\int f(x) \cdot g'(x)\ dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)\,dx}$$

<u>Strategy</u>: Try to choose f and g' such that:

- f is simpler when differentiated
- g' is simpler when integrated
- $\int f'(x) \cdot g(x)\,dx$ is simpler to integrate than $\int f(x) \cdot g'(x)\,dx$

<u>Another way to write it:</u>

$$u = f(x) \quad \rightarrow \quad \frac{du}{dx} = f'(x) \quad \rightarrow \quad du = f'(x)dx$$

$$v = g(x) \quad \rightarrow \quad \frac{dv}{dx} = g'(x) \quad \rightarrow \quad dv = g'(x)dx$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Integration of Rational Functions $\int \frac{p(x)}{q(x)}dx$ Using Partial Fractions

1. **Check that degree(p) < degree(q).**

   If it's not, use polynomial division to rewrite p/q as a polynomial plus a new rational polynomial p_2/q_2 where degree(p_2) < degree(q_2).

   <u>Example:</u> $$\frac{x^3 - 4x - 10}{x^2 - x - 6} = x + 1 + \frac{3x - 4}{x^2 - x - 6}.$$

2. **Factor the denominator q until you can't anymore** (factor into "irreducible terms").

3. **Write p/q as a sum of partial fractions** (and solve for constants).

   <u>Examples</u>: $$\frac{x-1}{x^2+x} = \frac{x-1}{x(x+1)} = \frac{\boldsymbol{A}}{x} + \frac{\boldsymbol{B}}{x+1}$$

   $$\frac{2x-3}{x^3+x} = \frac{2x-3}{x(x^2+1)} = \frac{\boldsymbol{A}}{x} + \frac{\boldsymbol{B}x + \boldsymbol{C}}{x^2+1}$$

   $$\frac{2x^3+5x-1}{(x+1)^3(x^2+1)^2} = \frac{\boldsymbol{A}}{x+1} + \frac{\boldsymbol{B}}{(x+1)^2} + \frac{\boldsymbol{C}}{(x+1)^3} + \frac{\boldsymbol{D}x + \boldsymbol{E}}{x^2+1} + \frac{\boldsymbol{F}x + \boldsymbol{G}}{(x^2+1)^2}$$

4. **Integrate!** (It should be easier now.)