Fundamental Theorem of Calculus, Riemann Sums, Substitution Integration Methods

104003 Differential and Integral Calculus I
Technion International School of Engineering 2010-11
Tutorial Summary – February 27, 2011 – Kayla Jacobs

Indefinite vs. Definite Integrals

> A calc student upset as could be
> That his antiderivative didn't agree
> With the one in the book
> E'en aft one more look.
> Oh! Seems he forgot to write the "+ C".
> *-Anonymous*

- **Indefinite integral:** $\int f(x)\,dx$

  The **function** *F*(x) that answers question:
  "What function, when differentiated, gives *f*(x)?"

- **Definite integral:** $\int_a^b f(x)\,dx$
  - The **number** that represents the area under the curve *f*(x) between *x*=*a* and *x*=*b*
  - *a* and *b* are called the **limits of integration**.
  - Forget the +c. Now we're calculating actual values .

Fundamental Theorem of Calculus (Relationship between definite & indefinite integrals)

If $F(x) \coloneqq \int_a^x f(t)\,dt$ and *f* is continuous, then *F* is differentiable and $F'(x) = f(x)$.

Important Corollary: For any function *F* whose derivative is f (i.e., $F'(x) = f(x)$),

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

This lets you easily calculate definite integrals!

Definite Integral Properties

- $\int_a^a f(x)\,dx = 0$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ whether or not $c \in [a, b]$

Area in [a,b] Bounded by Curve f(x)

Case 1: Curve entirely above x-axis. *Really easy!* Area = $\int_a^b f(x)\,dx$

Case 2: Curve entirely below x-axis. *Easy!* Area = $\left|\int_a^b f(x)\,dx\right|$ = $-\int_a^b f(x)\,dx$

Case 3: Curve sometimes below, sometimes above x-axis. *Sort of easy!* Break up into sections.

Average Value

The average value of function f(x) in region [a,b] is:

$$\text{average} = \frac{\int_a^b f(x)\,dx}{b-a}$$

Riemann Sum

Let **[a,b]** = closed interval in the domain of function *f*

Partition [a,b] into n subdivisions: { $[x_0,x_1]$, $[x_1,x_2]$, $[x_2,x_3]$, … , $[x_{n-1},x_n]$} where **a** = $x_0 < x_1 < \ldots < x_{n-1} < x_n$ = **b**

The **Riemann sum** of function f over interval [a,b] is:

$$S = \sum_{i=1}^{n} f(y_i) \cdot (x_i - x_{i-1})$$

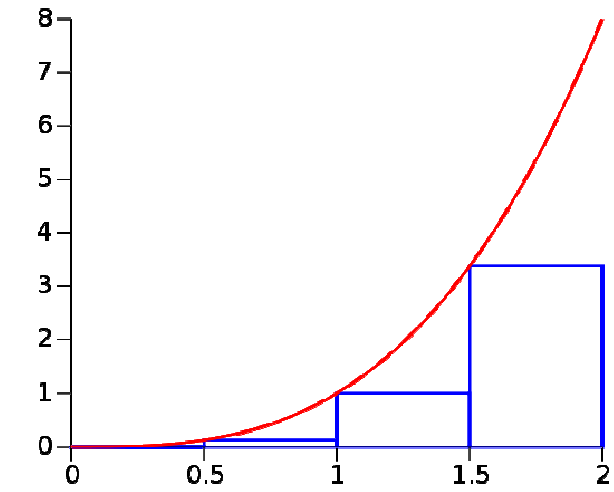
where y_i is *any* value between x_{i-1} and x_i. Note $(x_i - x_{i-1})$ is the length of the i^{th} subdivision $[x_{i-1}, x_i]$.

If for all *i*:	**then…**
$y_i = x_{i-1}$	S = **Left** Riemann sum
$y_i = x_i$	S = **Right** Riemann sum
$y_i = (x_i + x_{i-1})/2$	S = **Middle** Riemann sum
$f(yi) = (f(x_{i-1}) + f(x_i))/2$	S = **Trapezoidal** Riemann sum
$f(y_i)$ = maximum of *f* over $[x_{i-1}, x_i]$	S = **Upper** Riemann sum
$f(y_i)$ = minimum of *f* over $[x_{i-1}, x_i]$	S = **Lower** Riemann sum

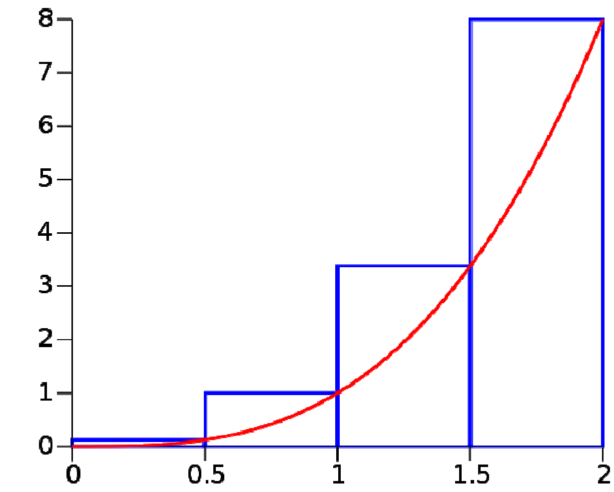
As $n \to \infty$, ***S* converges to the value of the definite integral** of *f* over [*a*,*b*]: $\lim_{n\to\infty} S = \int_a^b f(x)\,dx$

Ex: Riemann sum methods of *f*(*x*) = *x*³ over interval [*a*, *b*] = [0, 2] using 4 equal subdivisions of 0.5 each:

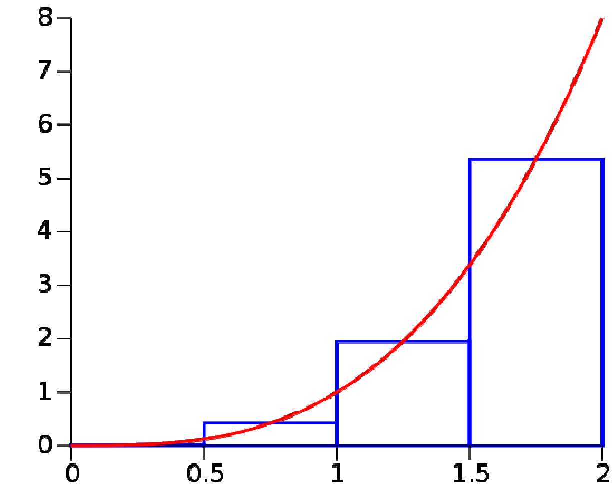
(1) Left Riemann sum:



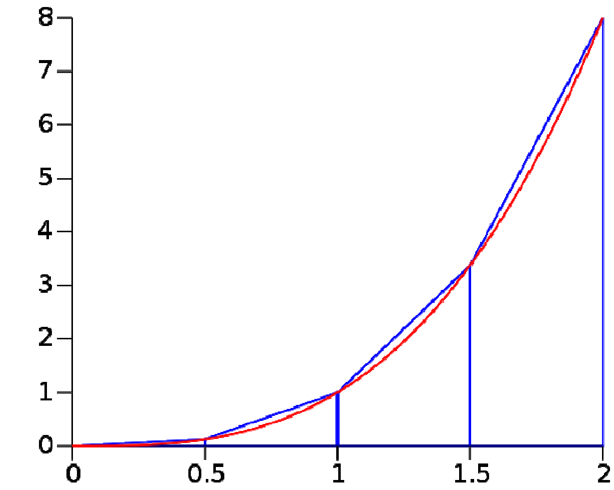
(2) Right Riemann sum:



(4) Middle Riemann sum:



(3) Trapezoidal Riemann sum:



Integration Method: u-substitution

$$\int_a^b f(u(x)) \cdot u'(x)\,dx = \int_{u(a)}^{u(b)} f(u(x))\,du$$

…where $du = u'(x)dx$ (because $u'(x) = du/dx$).

Notes:

- This is basically derivative chain rule in reverse.
- The hard part is figuring out what a good *u* is.
- If it's a **definite integral**, **don't forget to change the limits of integration!** $a \to u(a), \quad b \to u(b)$
- If it's an **indefinite integral, don't forget to change back to the original variable** at the end, and **+c**.

Basic Trigonometric Derivatives and Indefinite Integrals

$D(\sin x) = \cos x$ $\qquad \int \cos x \; dx = \sin x + C$

$D(\cos x) = -\sin x$ $\qquad \int \sin x \; dx = -\cos x + C$

$D(\tan x) = \sec^2 x$ $\qquad \int \sec^2 x \; dx = \tan x + C$

$D(\cot x) = -\csc^2 x$ $\qquad \int \csc^2 x \; dx = -\cot x + C$

$D(\sec x) = \sec x \tan x$ $\qquad \int \sec x \tan x \; dx = \sec x + C$

$D(\csc x) = -\csc x \cot x$ $\qquad \int \csc x \cot x \; dx = -\csc x + C$

From trigonometric identities and u-substitution:

$$\int \tan x \; dx = \ln|\sec x| + C$$

$$\int \cot x \; dx = \ln|\sin x| + C$$

$$\int \sec x \; dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x \; dx = \ln|\csc x - \cot x| + C$$

Integration Method: Trigonometric Substitution for Rational Functions of Sine and Cosines

To integrate a rational function of sin(x) and cos(x), try the substitution:

$$t = \tan(\frac{x}{2}) \qquad dx = \frac{2}{1+t^2}dt$$

Use the following trig identities to transform the function into a rational function of *t*:

$$\sin(x) = \frac{2\tan(\frac{x}{2})}{1+\tan^2(\frac{x}{2})} = \frac{2t}{1+t^2}$$

$$\cos(x) = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)} = \frac{1-t^2}{1+t^2}$$

$$\tan(x) = \frac{2\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)} = \frac{2t}{1-t^2}$$

Integration Method: Trigonometric Substitution

If the integral involves…	… then substitute…	… and use the trig identity…
$\boldsymbol{a^2 - u^2}$	$u = a \cdot \sin\theta$	$1 - \sin^2\theta = \cos^2\theta$
$\boldsymbol{a^2 + u^2}$	$u = a \cdot \tan\theta$	$1 + \tan^2\theta = \sec^2\theta$
$\boldsymbol{u^2 - a^2}$	$u = a \cdot \sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

Steps:

1. Notice that the integral involves one of the terms above.
2. Substitute the appropriate u. Make sure to change the dx to a du (with relevant factor).
3. Simplify the integral using the appropriate trig identity.
4. Rewrite the new integral in terms of the original non-*Θ* variable (draw a reference right-triangle to help).
5. Solve the (hopefully now much easier) integral,