A stress integration algorithm for J_3-dependent elasto-plasticity models

X. Wang [a,*], K.J. Bathe [b], J. Walczak [a]

[a] ADINA R&D, Inc., 71 Elton Avenue, Watertown, MA 02472, USA
[b] Massachusetts Institute of Technology, Mechanical Engineering Department, Cambridge, MA 02139, USA

Abstract

In this paper, a stress integration algorithm is presented for a generalized elasto-plastic material model governed by the three stress invariants I_1, J_2 and J_3. The methodology is successfully applied to the Mohr–Coulomb material model with a non-associated flow rule and implemented in ADINA.

Keywords: Elasto-plasticity; Stress integration algorithm; Mohr–Coulomb yield criterion

1. Introduction

In the finite element method, the integration scheme of the inelastic constitutive behavior directly controls the accuracy and stability of the overall numerical solution [1]. So far, effective methodologies have been proposed for plasticity models whose yield criteria can be written as functions of I_1 (the first stress invariant) and J_2 (the second deviatoric stress invariant) [1] [2]. However, no efficient algorithms are available for general material models of great engineering interests, in which not only I_1 and J_2, but also the third deviatoric stress invariant J_3 is used, such as in the Mohr–Coulomb material model. In this paper, an integration scheme is proposed for such models and specifically for the Mohr–Coulomb material description. The scheme is based on return mapping of the stresses. The objective of this paper is to briefly present the algorithm as implemented in ADINA and give some results obtained in the analysis of the excavation of a set of twin tunnels.

2. The algorithm

An inelastic solution scheme requires two ingredients: the stress integration and the calculation of the consistent tangent stress–strain matrix [1].

* Corresponding author. Tel.: +1 (617) 926-5199; Fax: +1 (617) 926-0238; E-mail: jay@adina.com

Computational Fluid and Solid Mechanics
K.J. Bathe (Editor)

2.1. The stress integration scheme

During the stress calculations, for any given strain increment $\Delta\mathbf{e}$, the corresponding stress increment must be computed iteratively. The stresses at time $t + \Delta t$ can be written in the following form

$$ {}^{t+\Delta t}\boldsymbol{\sigma} = \mathbf{C}^{E}\,{}^{t+\Delta t}\mathbf{e}^{E} = \mathbf{C}^{E}\left({}^{t+\Delta t}\mathbf{e} - {}^{t}\mathbf{e}^{P} - \Delta\mathbf{e}^{P}\right) \tag{1} $$

where $\mathbf{C}^E$ is the elastic stress–strain matrix, ${}^{t+\Delta t}\mathbf{e}$ are the total strains at time $t + \Delta t$, ${}^{t+\Delta t}\mathbf{e}^E$ are the total elastic strains at time $t + \Delta t$, ${}^{t}\mathbf{e}^P$ are the total plastic strains at time t, and $\Delta\mathbf{e}^P$ are the total plastic strain increments. For a general elasto-plastic material model with its plastic potential function written as $g = g(I_1, J_2, J_3, H_\alpha)$, where the H_α are the N state variables ($\alpha = 1, 2, \ldots N$), the plastic strain increments are given by

$$ \Delta e_{ij}^{P} = \Delta e_{m}^{P}\delta_{ij} + \Delta e_{ij}^{\prime P} \quad (i, j = 1, 2, 3) \tag{2} $$

where δ_{ij} is the Kronecker delta. Using the Euler backward method, the above mean plastic strain and deviatoric plastic strain increments based on the flow rule are, respectively

$$ \Delta e_{m}^{P} = \frac{1}{3}\Delta e_{ii}^{P} = \frac{1}{3}\,{}^{t+\Delta t}\!\left(\frac{\partial g}{\partial p}\right)\Delta\lambda \tag{3} $$

$$ \Delta e_{ij}^{\prime P} = {}^{t+\Delta t}\!\left(\frac{\partial g}{\partial S_{ij}}\right)\Delta\lambda - \frac{2}{3}\,{}^{t+\Delta t}\!\left(\frac{\partial g}{\partial J_3}\right){}^{t+\Delta t}J_2\Delta\lambda\delta_{ij} \tag{4} $$

and the pressure and deviatoric stresses at time $t + \Delta t$ are

$$ {}^{t+\Delta t}p = {}^{t+\Delta t}p^{E} + \frac{\Delta\lambda}{3a_M}\,{}^{t+\Delta t}\!\left(\frac{\partial g}{\partial p}\right) \tag{5} $$

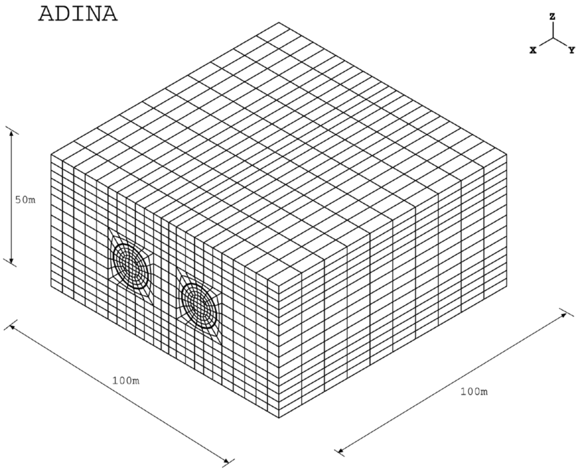

Fig. 1. 3-D finite element mesh for the twin tunnels.

$$ {}^{t+\Delta t}S_{ij} = {}^{t+\Delta t}S^{E}_{ij} - \frac{1}{a_E}\,{}^{t+\Delta t}\left(\frac{\partial g}{\partial S_{ij}}\right)\Delta\lambda + \frac{2}{3a_E}\,{}^{t+\Delta t}\left(\frac{\partial g}{\partial J_3}\right){}^{t+\Delta t}J_2\Delta\lambda\delta_{ij} \qquad (6) $$

where pressure p is defined as $-I_1/3$, p^E and S^E_{ij} are the elastic predictors for p and S_{ij}, respectively, a_E and a_M are elastic constants, $\Delta\lambda$ is called the plastic multiplier and the superscript $t+\Delta t$ denotes the time of solution [1]. It is noted that ${}^{t+\Delta t}\left(\frac{\partial g}{\partial p}\right)$, ${}^{t+\Delta t}\left(\frac{\partial g}{\partial S_{ij}}\right)$ and ${}^{t+\Delta t}\left(\frac{\partial g}{\partial J_3}\right)$ are functions of the stress components ${}^{t+\Delta t}p$, ${}^{t+\Delta t}S_{ij}$ and the state variables ${}^{t+\Delta t}H_\alpha$. Therefore, ${}^{t+\Delta t}p$ and ${}^{t+\Delta t}S_{ij}$ cannot be obtained explicitly from Eqs. (5) and (6), which represents some difficulties in the stress integration with the potential function $g\left({}^{t+\Delta t}I_1, {}^{t+\Delta t}J_2, {}^{t+\Delta t}J_3, {}^{t+\Delta t}H_\alpha\right)$ and the yield function

$$ f = f({}^{t+\Delta t}I_1, {}^{t+\Delta t}J_2, {}^{t+\Delta t}J_3, {}^{t+\Delta t}H_\alpha) \qquad (7) $$

In general, the increments in the state variables can be written as functions of the stresses, plastic strain increments, and the state variables ${}^{t+\Delta t}H_\beta$ themselves

$$ \Delta H_\alpha = \Delta h({}^{t+\Delta t}I_1, {}^{t+\Delta t}J_2, {}^{t+\Delta t}J_3, \Delta e^p_m, \Delta e'^p_{ij}, {}^{t+\Delta t}H_\beta) \quad (\alpha, \beta = 1, 2, \ldots N) \qquad (8) $$

For a 3-dimensional problem, if ΔH_α, or ${}^{t+\Delta t}H_\alpha$, can be expressed explicitly in terms of the stress components and plastic strain increments, we need to solve Eqs. (5), (6) and (7) for a total of 8 primary unknowns at time $t+\Delta t$: one pressure component ${}^{t+\Delta t}p$, six deviatoric stress components ${}^{t+\Delta t}S_{ij}$, and one plastic multiplier $\Delta\lambda$. The solution is obtained using Newton–Raphson iterations.

2.2. Determination of the consistent tangential moduli

In a full Newton–Raphson scheme used to perform the global equilibrium iterations, the tangent constitutive relation at time $t+\Delta t$, consistent with the stress integration scheme, needs to be calculated. The constitutive relation is defined as the variation in the stresses ${}^{t+\Delta t}\boldsymbol{\sigma}$ as a consequence of a variation in the total strains ${}^{t+\Delta t}\mathbf{e}$

$$ {}^{t+\Delta t}\mathbf{C}^{EP} = \frac{\partial\,{}^{t+\Delta t}\boldsymbol{\sigma}}{\partial\,{}^{t+\Delta t}\mathbf{e}} \qquad (9) $$

The components for the constitutive tensor take the form

$$ {}^{t+\Delta t}C^{EP}_{ijkl} = \frac{\partial\,{}^{t+\Delta t}\sigma_{ij}}{\partial\,{}^{t+\Delta t}e_{kl}} = -\frac{\partial\,{}^{t+\Delta t}p}{\partial\,{}^{t+\Delta t}e_{kl}} + \frac{\partial\,{}^{t+\Delta t}S_{ij}}{\partial\,{}^{t+\Delta t}e_{kl}} \quad (i = j) \qquad (10) $$

$$ {}^{t+\Delta t}C^{EP}_{ijkl} = \frac{\partial\,{}^{t+\Delta t}\sigma_{ij}}{\partial\,{}^{t+\Delta t}e_{kl}} = \frac{\partial\,{}^{t+\Delta t}S_{ij}}{\partial\,{}^{t+\Delta t}e_{kl}} \quad (i \neq j) \qquad (11) $$

The expressions $\frac{\partial\,{}^{t+\Delta t}p}{\partial\,{}^{t+\Delta t}e_{kl}}$ and $\frac{\partial\,{}^{t+\Delta t}S_{ij}}{\partial\,{}^{t+\Delta t}e_{kl}}$ are obtained by the differentiation of the yield function, equations to calculate the stresses, and equations expressing the hardening relations with respect to the total strain components. Removing the left superscripts $t+\Delta t$ from all the variables for ease of writing, the differentiations are written as follows

$$ \frac{\partial f}{\partial p}\frac{\partial p}{\partial e_{kl}} + \frac{\partial f}{\partial J_2}\frac{\partial J_2}{\partial S_{mn}}\frac{\partial S_{mn}}{\partial e_{kl}} + \frac{\partial f}{\partial J_3}\frac{\partial J_3}{\partial S_{mn}}\frac{\partial S_{mn}}{\partial e_{kl}} + \frac{\partial f}{\partial H_\alpha}\frac{\partial H_\alpha}{\partial e_{kl}} = 0 \qquad (12) $$

$$ \frac{\partial p^E}{\partial e_{kl}} = \left(1 - \frac{\Delta\lambda}{3a_M}\frac{\partial^2 g}{\partial p^2}\right)\frac{\partial p}{\partial e_{kl}} - \frac{\Delta\lambda}{3a_M}\frac{\partial^2 g}{\partial p\partial S_{mn}}\frac{\partial S_{mn}}{\partial e_{kl}} - \frac{1}{3a_M}\frac{\partial g}{\partial p}\frac{\partial\Delta\lambda}{\partial e_{kl}} - \frac{\Delta\lambda}{3a_M}\frac{\partial^2 g}{\partial p\partial H_\alpha}\frac{\partial H_\alpha}{\partial e_{kl}} \qquad (13) $$

$$ \begin{aligned} \frac{\partial S^E_{ij}}{\partial e_{kl}} = {} & \frac{\Delta\lambda}{a_E}\left(\frac{\partial^2 g}{\partial S_{ij}\partial p} - \frac{2}{3}J_2\frac{\partial^2 g}{\partial J_3\partial p}\delta_{ij}\right)\frac{\partial p}{\partial e_{kl}} + \frac{\partial S_{ij}}{\partial e_{kl}} \\ & + \frac{\Delta\lambda}{a_E}\left(\frac{\partial^2 g}{\partial S_{ij}\partial S_{mn}} - \frac{2}{3}\frac{\partial g}{\partial J_3}S_{mn}\delta_{ij} - \frac{2}{3}J_2\frac{\partial^2 g}{\partial J_3\partial S_{mn}}\delta_{ij}\right)\frac{\partial S_{mn}}{\partial e_{kl}} \\ & + \frac{1}{a_E}\left(\frac{\partial g}{\partial S_{ij}} - \frac{2}{3}J_2\frac{\partial g}{\partial J_3}\delta_{ij}\right)\frac{\partial\Delta\lambda}{\partial e_{kl}} \\ & + \frac{\Delta\lambda}{a_E}\left(\frac{\partial^2 g}{\partial S_{ij}\partial H_\alpha} - \frac{2}{3}J_2\frac{\partial^2 g}{\partial J_3\partial H_\alpha}\delta_{ij}\right)\frac{\partial H_\alpha}{\partial e_{kl}} \end{aligned} \qquad (14) $$

In the above equations, i, j, k, l, m and n range from 1 to 3. The differentiation of each equation governing the strain hardening of the state variables, expressed as $F_H = 0$, with respect to the strain components gives

$$ \frac{\partial F_H}{\partial p}\frac{\partial p}{\partial e_{kl}} + \frac{\partial F_H}{\partial S_{mn}}\frac{\partial S_{mn}}{\partial e_{kl}} + \frac{\partial F_H}{\partial\Delta\lambda}\frac{\partial\Delta\lambda}{\partial e_{kl}} + \frac{\partial F_H}{\partial H_\alpha}\frac{\partial H_\alpha}{\partial e_{kl}} = 0 \quad (\alpha = 1, 2, \ldots N) \qquad (15) $$

Combining Eqs. (12), (13), (14) and (15), we have $(8+N)$ equations that are established using the stress integration algorithm and solved for the unknowns: $\partial\Delta\lambda/\partial e_{kl}$, $\partial p/\partial e_{kl}$, $\partial S_{ij}/\partial e_{kl}$, and $\partial H_\alpha/\partial e_{kl}$.

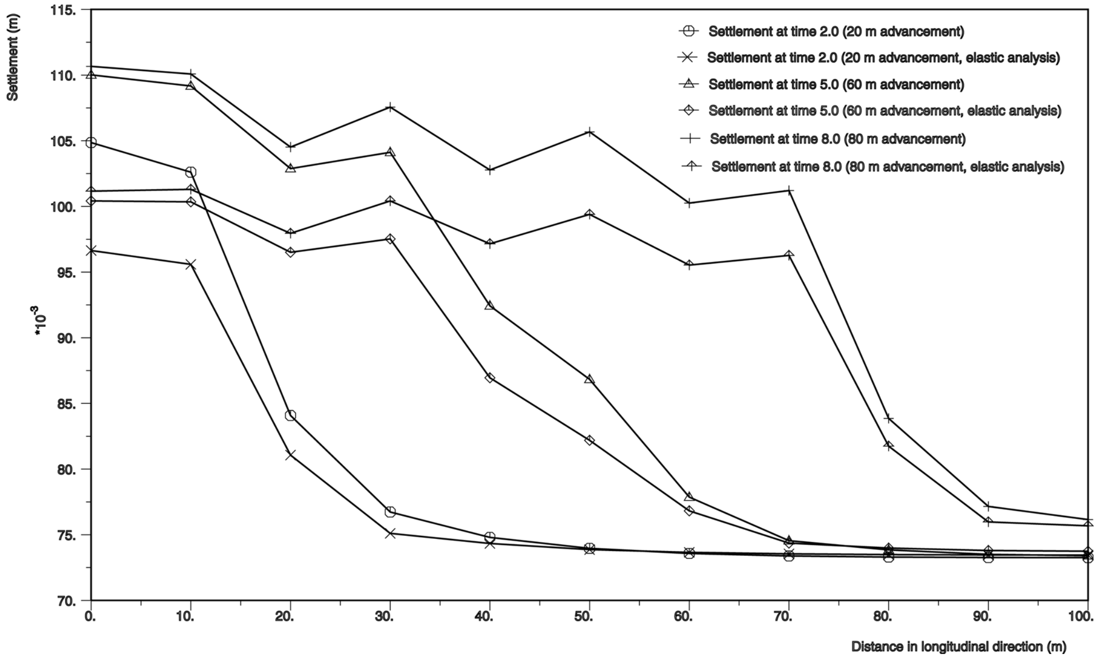

Fig. 2. Vertical displacement distribution along the tunnel crown at different excavation stages.

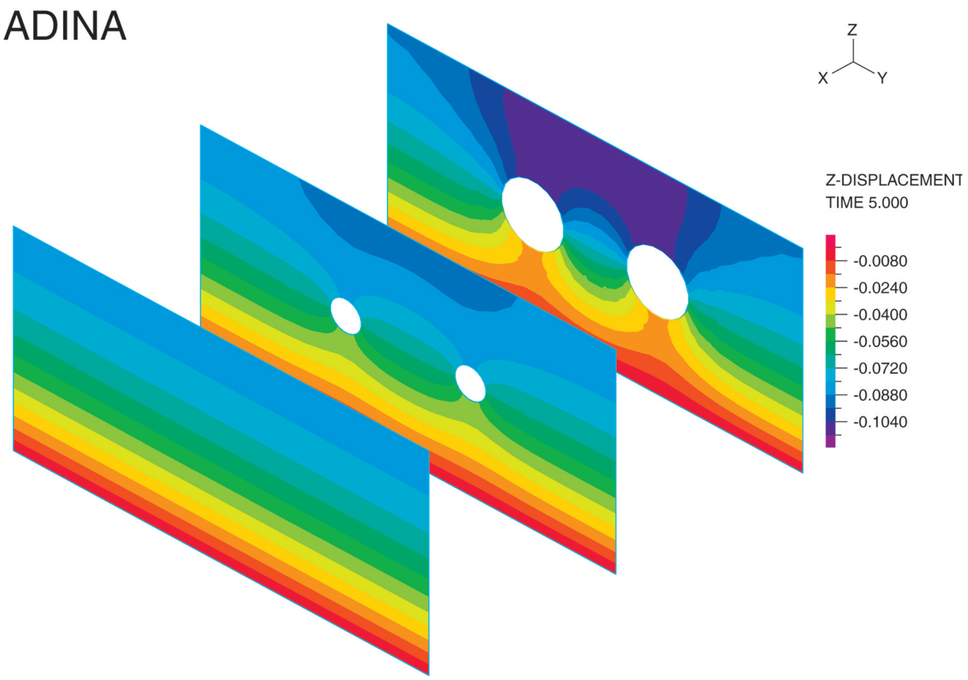

Fig. 3. Vertical displacement distribution at the sections $X = 0$, 30 and 60 m across the tunnel axis (m).

3. Application to the Mohr–Coulomb model

The yield function and the potential function of the Mohr–Coulomb material model are [3]

$$f = I_1 \sin\phi + \frac{1}{2}\big[3(1-\sin\phi)\sin\theta + \sqrt{3}\,(3+\sin\phi)\cos\theta\big]\sqrt{J_2} - 3c\cos\phi \tag{16}$$

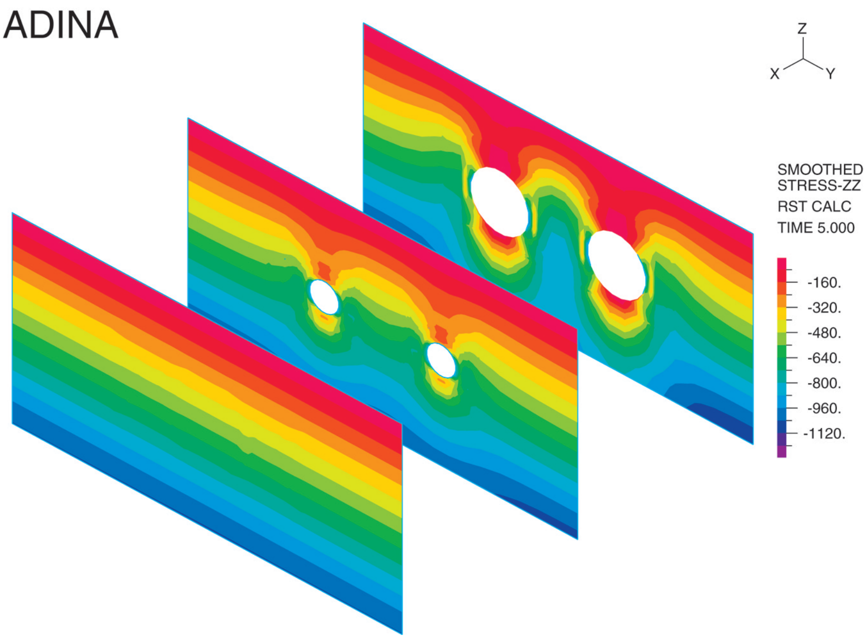

Fig. 4. Vertical normal stress distribution at the sections $X = 0$, 30 and 60 m across the tunnel axis (kPa).

$$g = I_1 \sin\psi + \frac{1}{2}\big[3(1-\sin\psi)\sin\theta + \sqrt{3}\,(3+\sin\psi)\cos\theta\big]\sqrt{J_2} - 3c\cos\psi \tag{17}$$

in which ϕ is the internal friction angle, ψ is the dilation angle, c denotes the material cohesion and θ is the Lode angle. The proposed return mapping algorithm for the Mohr–Coulomb model has been implemented in ADINA, and we give here some results obtained in a simulation of the sequential excavation of a set of twin tunnels constructed in a soft rock layer, see Fig. 1. Each tunnel was 15.6 m in diameter. A soft rock domain of $100 \times 100 \times 50$ m^3 was taken for the analysis. The soft rock was assumed to correspond to the following material parameters: $E = 240$ MPa, $\nu = 0.3$, unit weight $\gamma = 21.5$ kN/m^3, $\phi = 22.0°$, $\psi = 10.0°$, $c = 0.1$ MPa. The liner was modeled assuming a linear elastic material with $E = 5000$ MPa, $\nu = 0.25$, and unit weight $\gamma = 25.5$ kN/m^3. Five incremental excavation stages performed together for each tunnel were completed in the longitudinal direction, each incremental excavation comprising a 20 m advancement and containing 2 time steps to perform the radial excavation. The liner was installed right after each longitudinal excavation.

To obtain three-digit accuracy in energy values, convergence was reached using a maximum of 3 iterations in each time step. Fig. 2 shows the vertical displacement distribution at different excavation steps. The displacements are plotted for the nodal points along the tunnel crown. The displacements are small before the tunnel face approaches the cross section, but increase immediately after the face has passed, and then increase further, as the face progresses, until a level of about 11 cm. The displacement results assuming elastic conditions are also shown. These values are of course smaller. Similar conclusions can be drawn from Fig. 3, which shows the vertical displacement distribution at three sections at the time 5.0. Fig. 4 gives the corresponding vertical normal stress distributions.

4. Conclusions

A stress integration procedure has been presented for a general elasto-plastic material model in which the stress invariants I_1, J_2 and J_3 are relevant. The proposed formulation is capable of accommodating arbitrary yield criteria, flow rules and hardening laws provided, of course, the first and second derivatives of the yield and potential functions with respect to the stress components are available. The algorithm has been implemented for the Mohr–Coulomb material model with a non-associated flow rule.

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