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Component mode synthesis with subspace iterations for controlled accuracy of frequency and mode shape solutions

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ABSTRACT

The objective in this paper is to present an approach to improve component mode synthesis solutions using subspace iterations to obtain frequency and mode shape predictions of controlled accuracy. In traditional component mode synthesis analyses, the calculated frequencies and mode shapes are approximations of the exact frequencies and mode shapes of the finite element model, the error is unknown, may be large, and is usually not assessed. In the approach given here, the error is assessed and can be reduced to the desired level. The Craig–Bampton component mode synthesis is used, but the solution approach is also directly applicable to any other component mode synthesis scheme. Some example solutions are given to illustrate the use and the effectiveness of the solution approach.

1. Introduction

Component mode synthesis solution techniques [1,2] are widely used in finite element analyses. The solution approach was proposed long ago by Hurty [3], but when seen as a domain decomposition technique can in fact be traced back to much earlier work, see Ref. [4]. These methods can be effective for the solution of frequencies and mode shapes when complex structures are analyzed, as, for example, in the airplane industries. The representation of the complete structure is based on using large substructures. These individual structures, or components, are typically first analyzed in detail for their frequencies and mode shapes, and dynamic response, by different analysis groups. Then the complete structure is considered as an assemblage of the components, which can lead to a very large finite element system. However, usually only the lowest p frequencies and corresponding mode shapes of the complete finite element system with n degrees of freedom are needed, where $p \ll n$. The basic approach of a component mode synthesis analysis is to use the mode shape solutions of the individual components, i.e. substructures, to obtain approximations to the exact p frequencies and mode shapes of the complete model [2]. An important such procedure used is the Craig–Bampton method [5].

Since the component mode synthesis procedures only give approximate solutions to the exact eigenvalues and vectors (frequencies and mode shapes) of the complete structural model, it can be important to have some error measure, see Refs. [4,6] and the many references therein, to ensure the reliability of a response prediction. The scheme should give an approximation to the error in the solution when compared to the exact solution, which of course is unknown. Considering that the calculated eigenvalues are always larger than the exact values [2], the error measure should ideally give a proven upper bound to the exact error, be close to the exact error, and should be inexpensive to compute. Also, if the error is too large, a simple scheme to reduce the error should be available. This approach is valuable because in today's finite element analysis practice, a strong emphasis should be on the reliability of the numerical solutions rather than merely on computational effectiveness. The reliability should indeed be a requirement in the solution of the required frequencies and mode shapes, also because large systems of finite element equations can now be solved very accurately using the Lanczos transformation method and the Bathe subspace iteration technique [2].

As is well known, component mode synthesis solutions are closely related to Rayleigh–Ritz analysis and hence also to a subspace iteration. It follows therefore that the error measure used in the Bathe subspace iteration method may also be attractive for use in a component mode synthesis solution [2,7–9].

In recent work, Yin et al. [10] presented automated multilevel substructuring (AMLS) techniques using also the Bathe subspace iteration method with an error indicator. However, the emphasis in the following sections is on using a proven error bound and a simple scheme to decrease the error, when so wanted, monotonically to

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the desired value. This approach is used to obtain reliable solution results.

In the following sections, we review the Craig–Bampton component mode synthesis analysis and show how, in a natural further step, the Bathe subspace iteration method can be used to measure and reduce the error. While we focus on the Craig–Bampton technique, other methods can also be used with the given approach and error measure. Some illustrative example solutions are included to show the applicability of the solution approach.

2. The Craig–Bampton method

We consider the eigenvalue problem

$$\mathbf{K}\boldsymbol{\varphi} = \lambda\mathbf{M}\boldsymbol{\varphi} \tag{1}$$

where $\mathbf{K}$ and $\mathbf{M}$ are the symmetric stiffness and mass matrices of the complete finite element system of n equations. We seek the smallest p eigenvalues and corresponding eigenvectors, with $p \ll n$. Without loss of generality, we assume that the stiffness and mass matrices $\mathbf{K}$ and $\mathbf{M}$ are positive definite, and hence the eigenvalues we seek are

$$0 < \lambda_1 \leqslant \lambda_2 \cdots \leqslant \lambda_{p-1} \leqslant \lambda_p \tag{2}$$

with the corresponding $\mathbf{M}$-orthonormal eigenvectors $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_p$. For an unsupported structure we would simply apply a shift [2].

The Craig–Bampton method is a well-known component mode synthesis technique, and was designed when the calculation of the exact eigenvalues and vectors (or very close approximations thereof) of very large finite element models was technologically out of reach, or at least computationally very expensive to perform. The method is also naturally optimizing the workflow by using, in the solution of Eq. (1), the already calculated mode shapes of the components, i.e. of the substructures, that were already considered by different analysis groups [1–5].

Consider a generic finite element model as shown in Fig. 1. For the development of the governing equations, the dynamic equilibrium equations are partitioned as follows

$$\begin{bmatrix} \mathbf{M}_{bb} & \mathbf{M}_{bi} \\ \mathbf{M}_{ib} & \mathbf{M}_{ii} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_b \\ \ddot{\mathbf{u}}_i \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{bb} & \mathbf{K}_{bi} \\ \mathbf{K}_{ib} & \mathbf{K}_{ii} \end{bmatrix} \begin{bmatrix} \mathbf{u}_b \\ \mathbf{u}_i \end{bmatrix} = \begin{bmatrix} \mathbf{R}_b \\ \mathbf{R}_i \end{bmatrix} \tag{3}$$

where the entries in $\mathbf{u}_b$ refer to the r degrees of freedom of the boundary nodes and the entries in $\mathbf{u}_i$ correspond to the $(n-r)$ degrees of freedom of the interior (non-boundary) nodes. In the Craig–Bampton procedure the following transformation is used [5]

$$\begin{bmatrix} \mathbf{u}_b \\ \mathbf{u}_i \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \boldsymbol{\Phi}_c & \boldsymbol{\Phi}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_b \\ \mathbf{q} \end{bmatrix} \tag{4}$$

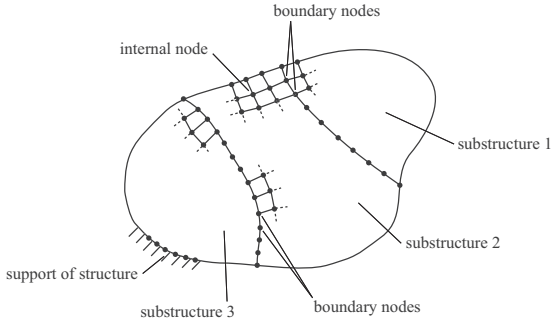

Fig. 1. Generic finite element model using substructuring; the boundary nodes are the nodes shared by substructures.

where $\mathbf{I}$ is the identity matrix, $\boldsymbol{\Phi}_n$ corresponds to s normal modes (corresponding to the smallest eigenvalues) with all boundary degrees of freedom fixed, usually $s \ll n$, $\boldsymbol{\Phi}_c$ is given as

$$\boldsymbol{\Phi}_c = -\mathbf{K}_{ii}^{-1}\mathbf{K}_{ib} \tag{5}$$

and $\mathbf{q}$ represents generalized displacements. The columns in $\boldsymbol{\Phi}_c$ represent static constraint modes, that is, the jth column represents the static displacements at all interior nodes when all boundary degrees of freedom are fixed except the jth boundary degree of freedom (corresponding to that column) is set to unity.

Hence the procedure uses the transformation matrix

$$\boldsymbol{\Psi} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \boldsymbol{\Phi}_c & \boldsymbol{\Phi}_n \end{bmatrix} \tag{6}$$

where $\boldsymbol{\Psi}$ is of dimension $n \times (r+s)$. Calculating as in a Rayleigh–Ritz solution [2]

$$\mathbf{K}_r = \boldsymbol{\Psi}^T\mathbf{K}\boldsymbol{\Psi} \tag{7}$$

$$\mathbf{M}_r = \boldsymbol{\Psi}^T\mathbf{M}\boldsymbol{\Psi} \tag{8}$$

the reduced eigenvalue problem is solved

$$\mathbf{K}_r\mathbf{X} = \mathbf{M}_r\mathbf{X}\boldsymbol{\rho} \tag{9}$$

The eigenvalue approximations are given in $\boldsymbol{\rho}$ and the eigenvector approximations are listed in $\overline{\boldsymbol{\Phi}}$ as

$$\overline{\boldsymbol{\Phi}} = \boldsymbol{\Psi}\mathbf{X} \tag{10}$$

Various modifications of this basic technique are of course possible, but the essence of all of the related procedures are the steps in Eqs. (7)–(10), with different transformation matrices [1–4].

3. The subspace iteration method

Here too we consider the eigenvalue problem in Eq. (1) and seek the solution of the p lowest eigenvalues and corresponding eigenvectors.

The subspace iteration procedure was developed to simultaneously solve for the required eigenpairs $(\lambda_i, \boldsymbol{\varphi}_i)$ accurately, see Refs. [2,7–9], and has been used abundantly in engineering and the sciences. The basic equations of the Bathe subspace iteration method are:

Pick q starting iteration vectors, $q > p$, to estabish $\mathbf{X}_0$ and then iterate with $k = 1, 2, 3, \ldots$

$$\mathbf{K}\overline{\mathbf{X}}_k = \mathbf{M}\mathbf{X}_{k-1} \tag{11}$$

$$\mathbf{K}_k = \overline{\mathbf{X}}_k^T\mathbf{K}\overline{\mathbf{X}}_k \tag{12}$$

$$\mathbf{M}_k = \overline{\mathbf{X}}_k^T\mathbf{M}\overline{\mathbf{X}}_k \tag{13}$$

$$\mathbf{K}_k\mathbf{Q}_k = \mathbf{M}_k\mathbf{Q}_k\boldsymbol{\Lambda}_k \tag{14}$$

$$\mathbf{X}_k = \overline{\mathbf{X}}_k\mathbf{Q}_k \tag{15}$$

until the following convergence tolerance is passed by all eigenvalues in $\boldsymbol{\Lambda}_k$ to be calculated [2,9]. Since [2,11]

$$\min_{\substack{i \\ \lambda_i \neq 0}} \left| \frac{\lambda_i - \lambda_i^{(k)}}{\lambda_i} \right| \leqslant \left[1 - \frac{\left(\lambda_i^{(k)}\right)^2}{\left(\mathbf{q}_i^{(k)}\right)^T\mathbf{q}_i^{(k)}} \right]^{1/2} \tag{16}$$

the convergence tolerance to be reached is

$$\left[1-\frac{\left(\lambda_i^{(k)}\right)^2}{\left(\mathbf{q}_i^{(k)}\right)^T\mathbf{q}_i^{(k)}}\right]^{1/2} \leq tol \tag{17}$$

with $\mathbf{q}_i^{(k)}$ the ith column in $\mathbf{Q}_k$. This convergence tolerance, *tol*, must be reached for all $i = 1, \ldots p$. The tolerance is set by the user as $tol = 10^{-2t}$, which means that the eigenvalues will be accurate to $2t$ digits and the vectors accurate to t digits. The error relation represents a *proven* upper bound, and is routinely used in the solution of very large eigenvalue problems [9].

However, when using the relation (17), it should be realized that while the relation represents upper bounds, a sufficiently small value of *tol* may be needed to see the actual accuracy of the calculated eigenvalues. Namely, we will always have $\lambda_j^{(k)} > \lambda_j$ but may also have that $\lambda_j^{(k)} > \lambda_{j+1}$ (by the minimax characterization of the calculated solutions [2]) with $\lambda_{j+1} > \lambda_j$. In this case it is possible that the error to λ_{j+1} is measured rather than to λ_j, as wanted, meaning that the actual error may be larger than the value given by the left-hand side of (17). Such cases can occur when the value of *tol* is not small enough, when the eigenvalues are closely spaced, and for the approximations of the larger eigenvalues. The cases could be identified using Sturm sequence checks at a considerable computational expense. However, they rarely occur if the number of iteration vectors is large and the value of *tol* is reasonably small ($t \leqslant 3$).

An important step is, of course, the selection of the starting iteration vectors in $\mathbf{X}_0$. As shown in Refs. [2,7,8], convergence is achieved provided the initial subspace is not $\mathbf{M}$-orthogonal to the eigenvectors sought. Also, the ultimate convergence rate of an iteration vector to the ith eigenvector is λ_i/λ_{q+1} where the iteration is carried out with q vectors when p eigenpairs are to be solved for. In today's practice an effective value for q to use is $q = \max\{p + 8, 2p\}$ [9].

Of course, the actual number of iterations required depends also on the quality of the starting iteration vectors, namely the closer the initial subspace (spanned by the starting vectors) is to the subspace sought, the smaller the number of iterations needed for convergence. Indeed, if the subspace of the starting vectors contains the eigenvectors to be calculated, these are directly obtained in the first subspace iteration. While in some analyses, good starting iteration vectors are naturally available from previous solutions, like in protein normal mode solutions [12] or design optimization problems [13], much research effort has been devoted to establish for more general cases good starting iteration vectors, see for example the references in Ref. [9]. Using a component mode synthesis procedure can be an effective way to proceed [2].

If we compare Eqs. (6)–(10) of the component mode synthesis solution with the subspace iteration method Eqs. (12)–(15), we recognize that $\overline{\mathbf{X}}_1$ can be thought of as $\mathbf{\Psi}$, and $\mathbf{X}_1$ can be thought of as $\overline{\mathbf{\Phi}}$. Of course, $\mathbf{X}_0$ is not established in the component mode synthesis solution.

Since Eqs. (6)–(10) correspond to Eqs. (12)–(15), the subspace iterations could now be carried out until convergence is reached to a prescribed value of *tol*, or only for a prescribed number of iterations. If at least one subspace iteration is performed, the error bound in (16) can directly be used [2]. While the subspace iteration is an expense, the additional computations are also frequently warranted because the solution accuracy can significantly increase as we demonstrate below. In addition, if the error is deemed too large, further subspace iterations can directly be performed until an acceptable error has been reached.

4. Illustrative solutions

The objective in this section is to give some example solutions that demonstrate the applicability of the schemes discussed above. In each example, prior to the study, we use the standard subspace iteration to solve very accurately for the required lowest p eigenvalues and vectors and then solve for approximations:

- first, using the Craig–Bampton scheme only,
- second, using the Craig–Bampton scheme followed by one subspace iteration with the calculation of the error measure applied to each eigenpair sought, and
- third, using a second subspace iteration to show the increase in accuracy obtained.

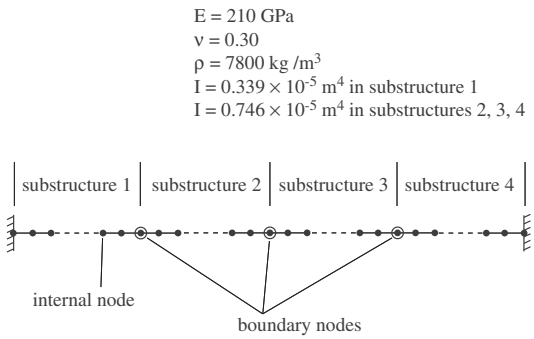

Fig. 2. Clamped–clamped beam (length 2 m) with 4 substructures and 3 boundary nodes.

4.1. Solution of a simple beam structure

We present this solution to demonstrate some detailed experiences with the solution approach discussed above.

The clamped–clamped beam structure shown in Fig. 2 is modeled using 100 three-dimensional beam elements and hence $n = 594$ (which corresponds to a very small finite element system). The number of frequencies and mode shapes required are 15, hence $p = 15$, and in the standard subspace iteration solution 30 iteration vectors are used, $q = 30$.

We use the 3 nodes in the interior of the beam at its quarter points as boundary nodes (hence the "boundary" means here the boundary between four equal substructures), see Fig. 2. Therefore, the Craig–Bampton scheme gives 18 static constraint modes, $r = 18$, and we use $s = 12$, with 3 eigenmodes from each substructure. In this way, $r + s = 30$, which is a good number of vectors to use for the subspace iteration [9]. Hence we set $q = 30$ also when performing the additional subspace iteration(s).

Table 1
Exact frequency, and percentage actual error and percentage error by error measure when using the component mode synthesis (CMS) only, and when using in addition one or two subspace iterations (SSI).

Exact freq. (cycles/s)	CMS % actual error in freq.	CMS + 1 SSI % actual error in freq.	CMS + 1 SSI % error by error measure	CMS + 2 SSI % actual error in freq.	CMS + 2 SSI % error by error measure
5.97724E+01	0.0074	0.0000	0.0001	0.0000	0.0000
1.21300E+02	0.0437	0.0000	0.0012	0.0000	0.0000
1.83706E+02	0.0370	0.0000	0.0005	0.0000	0.0000
2.33788E+02	0.2280	0.0000	0.0031	0.0000	0.0001
2.79778E+02	0.2967	0.0000	0.0400	0.0000	0.0013
3.35872E+02	0.3204	0.0000	0.1039	0.0000	0.0074
3.98557E+02	0.0356	0.0000	0.0368	0.0000	0.0028
4.62569E+02	0.5100	0.0000	0.3691	0.0000	0.0529
5.26707E+02	1.2506	0.0002	0.5699	0.0000	0.0643
5.90797E+02	0.0049	0.0000	0.0129	0.0000	0.0027
5.91374E+02	0.1584	0.0000	0.0004	0.0000	0.0000
6.53742E+02	4.1588	0.0041	2.0026	0.0002	0.3765
7.14822E+02	1.8018	0.0176	2.5880	0.0022	0.9172
7.66910E+02	13.7898	0.0525	5.5662	0.0048	1.4978
8.08075E+02	18.3176	0.1139	5.6747	0.0278	2.4001

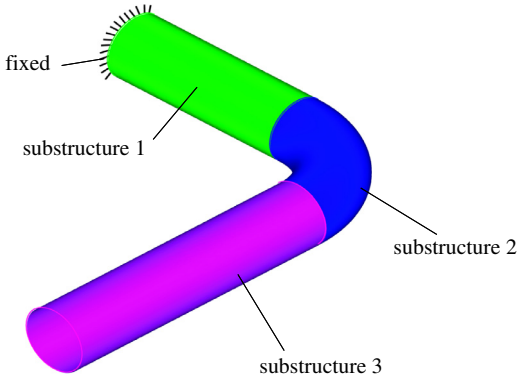

Fig. 3. Pipe structure modeled using shell elements.

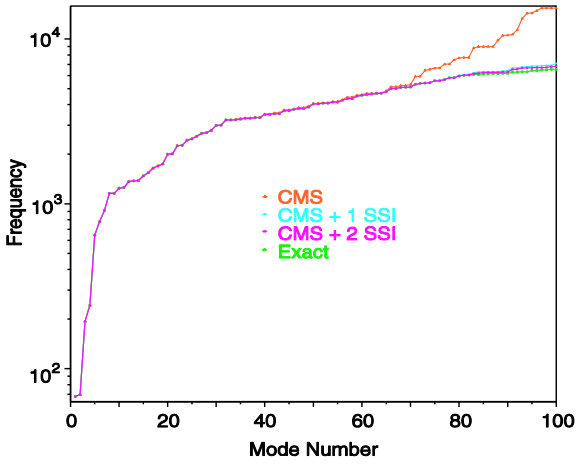

Fig. 4. Exact and predicted frequencies for pipe structure using the component mode synthesis (CMS) only, and one or two additional subspace iterations (SSI).

To measure the actual error for the 15 frequencies predicted using the Craig–Bampton scheme (with a total of 30 vectors) we calculate using the subspace iteration method the frequencies of the finite element model to good precision, using $2t = 12$ and the Sturm Sequence check [2].

The three eigenmodes corresponding to the lowest frequencies of each of the substructures for the Craig–Bampton scheme are also calculated using the subspace iteration method, and these we compute using $2t = 6$.

Table 1 gives the accurate 15 lowest frequencies, the *actual* percentage errors for the 15 frequencies when only the Craig–Bampton scheme is used, the *actual* percentage errors when one additional subspace iteration is used, and the error bounds (in percentage) calculated using the relation (16). It also includes the same information when a second subspace iteration is performed.

We see that the single additional subspace iteration significantly improves the accuracy of the solution and that the error measure approximates the actual error in a conservative manner.

4.2. Solution of a pipe structure modeled using shell elements

In this analysis we consider the pipe bend shown in Fig. 3 and seek to solve for the lowest 100 eigenvalues and corresponding eigenvectors, that is $p = 100$. We use a mesh of MITC4 shell elements leading to 258,900 degrees of freedom. In this case we use $q = 200$ with $r = 130$ and $s = 70$.

The same calculations as in Section 4.1 are performed. The results for the accurate frequencies, the frequencies calculated

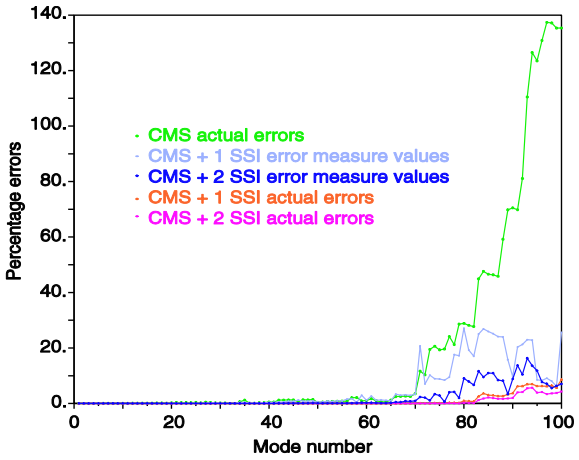

Fig. 5. Percentage errors in frequencies when using only the component mode synthesis (CMS), and one or two additional subspace iterations (SSI).

using the component mode synthesis scheme only, using this scheme with one additional subspace iteration and two additional subspace iterations are plotted in Fig. 4. The actual errors and the error bounds using the relation (16) (in percentage) are plotted in Fig. 5.

We see here too that the errors in the component mode synthesis solution are quite large for the higher frequencies and these errors are much reduced using the subspace iterations. Performing one subspace iteration reduces the largest error from about 135% to about 9%. The error measure is quite close to the actual error for the low frequencies but overestimates the error for the higher frequencies. Hence, a conservative approach to the solution is to perform more subspace iterations or use more iteration vectors, that is, increase q. Both example solutions, in Section 4.1 and this section, illustrate that indeed a better error measure, with less error overestimation, would still be desirable.

5. Concluding remarks

The objective in this paper was to present a scheme for usual component mode synthesis solutions, with which the errors in the frequency predictions can be conservatively estimated and then, if necessary, directly reduced. The scheme requires that at least one subspace iteration be carried out after the usual component mode synthesis solution. However, this subspace iteration is likely to also reduce the errors obtained in the traditional component mode synthesis analysis significantly. Furthermore, additional subspace iterations can directly be performed to obtain the desired accuracy.

Hence, we may also interpret the solution approach as using the component mode synthesis technique to construct an excellent starting subspace for the subspace iteration method when very large and complex structures are considered.

The theory was summarized and some example solutions were given that illustrate the use of the solution scheme. While the error measure is clearly valuable, the study also showed that a better error measure, that gives values closer to the exact errors, might still be looked for. But as mentioned above, this measure should ideally give good upper bounds on errors for the component mode synthesis and subspace iteration solutions, see for example Ref. [14] for a valuable development but where this has not been achieved as yet.

Further efforts to possibly reach improved solution techniques of frequencies and mode shapes are most valuable and can be pursued, for example, using automated multilevel substructuring or general group theoretic procedures, see Refs. [10,15] with the

references therein. Any proposed solution procedure should, however, be general and be evaluated versus efficient implementations of the widely used Lanczos transformation and Bathe subspace iteration methods that are employed with proven error bounds and lead to good solution accuracy. Namely, in engineering analysis the frequencies need frequently be calculated to about 6 digits accuracy, so that the spatial derivatives of the mode shapes are sufficiently accurate for the required stresses.

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