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Conserving energy and momentum in nonlinear dynamics: A simple implicit time integration scheme

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Abstract

We focus on a simple implicit time integration scheme for the transient response solution of structures when large deformations and long time durations are considered. Our aim is to have a practical method of implicit time integration for analyses in which the widely used Newmark time integration procedure is not conserving energy and momentum, and is unstable. The method of time integration discussed in this paper is performing well and is a good candidate for practical analyses.

Keywords: Nonlinear dynamics; Implicit time integration; Conservation of momentum and energy; Trapezoidal rule; Composite scheme

1. Introduction

Much research effort has been focused on the development of effective finite element methods for nonlinear dynamic analysis. An important part has been the development of step-by-step direct time integration schemes for the nonlinear equations of motion. For fast transient analyses, like crash simulations, the explicit central difference method is now widely used. However, this method is only conditionally stable and requires a very small time step for the response solution. Hence, for many problems, and in particular for structural vibration analyses, the use of an implicit technique can be much more effective [1].

Some of the first implicit time integration procedures used were the Houbolt, Newmark, and Wilson methods [1]. Of these, the Newmark method, and specifically the trapezoidal rule was soon recognized to be most effective and is now widely used in practical analyses. The method has been analyzed extensively, theoretically and in example solutions. The trapezoidal rule is in general effective, because it is a single step method, only establishes dynamic equilibrium at the discrete times considered, and is a second-order accurate procedure.

In the first nonlinear dynamic analyses using implicit direct time integration schemes the solutions were pursued without equilibrium iterations; that is, new tangent matrices were computed at the beginning of each step, and the solution was simply marched forward. However, it was soon recognized that establishing a tight dynamic equilibrium at the discrete time steps by iteration can be very important [2]. But even then, when large deformation dynamic analyses are pursued that require time integrations over relatively long time intervals, the trapezoidal rule can become unstable. This instability usually manifests itself in that the response grows, and energy and momentum are not conserved. This undesirable property was realized some time ago and various remedies and other single step time integration procedures were proposed. For a review of various research efforts, see Kuhl and Crisfield [3].

The research efforts can be classified, in essence, into three categories. In the first, new time integration procedures were proposed to introduce some numerical damping. A comprehensive such technique is the generalized alpha-method [4]. In the second category, Lagrange multipliers are employed to enforce momentum and energy

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conservation, and in the third category, conservation is enforced algorithmically, for example by using weighted stresses. We refer to Refs. [3–6] and the references therein for the many specific contributions.

In engineering practice, we would like to have a method that is accurate and efficient for many applications. The trapezoidal rule fulfills this requirement to a large extent except in large deformation analyses over long time periods. For these cases, we would like to have a method that operates much like the trapezoidal rule and ideally has these characteristics:

- The method solves the dynamic equilibrium equations 'at' the discrete time points of interest, with the externally applied loads, inertia forces, damping forces, nodal forces corresponding to the element stresses, and contact forces 'at' these times.
- The method is applicable to elastic as well as general inelastic analyses.
- The method does not involve additional variables, like Lagrange multipliers.
- The method has no parameter to choose or adjust, by the analyst, for specific analysis cases.
- The method is second-order accurate and remains stable even in large deformation and long time response solutions when the trapezoidal rule fails.
- Provided a reasonable time step size is used, chosen much like in linear analysis [1], the method gives an accurate response solution. If the time step is too large, the solution is of course inaccurate but does not become unstable.
- The time integration scheme is simple and computationally efficient using in the Newton–Raphson iterations only a symmetric 'effective' tangent stiffness matrix and 'effective' load vectors, like when using the trapezoidal rule.

While many techniques have been proposed, there is still a need for a method that has the above characteristics.

The objective of this paper is to present and discuss a procedure that is quite effective considering these desirable attributes. The method is a simple combination of the trapezoidal rule and the three-point backward Euler method. Of course, both these techniques have been around for a long time [7]. The two techniques were combined by Bank et al. to obtain a scheme for transient simulations of silicon devices and circuits [8]. This scheme referred to as the TRBDF2 method has found wide applications in solving differential equations outside the area of structural analysis, see for example Refs. [8–13]. Our purpose is to solve with this basic approach the second-order equations in time of nonlinear dynamic structural analysis. We have published the method for this application already in Refs. [14,15], and have used the method in ADINA already for some time. However, the scheme has not been exposed sufficiently for its effectiveness in certain nonlinear dynamic analyses. As we illustrate below, the method clearly deserves more attention since it solves for an accurate response when other used techniques fail or are not close to having the desired attributes listed above.

In this paper we once more present the method, but a major reason for the paper is to demonstrate the performance of the scheme, and discuss the method, considering some large deformation dynamic solutions. For this purpose, we specifically solve the test problems proposed by Kuhl and Crisfield [3], and discuss the various properties of the scheme with specific reference to the above desired characteristics. While the endeavor to develop more effective schemes should of course continue, the procedure given here does have some excellent and notable attributes.

2. The time integration scheme

The governing finite element equations to be solved are

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{F}(\mathbf{U}, \text{time}) = \mathbf{R}(\text{time}) \quad \text{(plus initial conditions)} \tag{1}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ is the damping matrix, $\mathbf{U}$ is the vector of nodal displacements/rotations, $\mathbf{F}$ denotes the vector of nodal point forces corresponding to the element internal stresses, $\mathbf{R}$ is the vector of externally applied nodal point forces/moments and an overdot denotes time derivative. We assume that the mass and damping matrices are constant, an assumption which can however be easily removed.

In the direct time integration, we aim to obtain the solution of Eq. (1) at discrete time points. Let us assume that the time step Δt is constant, and that the solution has been obtained up to time t; hence all the solution variables up to time t are known. The time-stepping scheme will give the solution for time $t + \Delta t$. The algorithm can then be used recursively to calculate the solutions at all discrete time points.

The approach used in the scheme is to calculate the unknown displacements, velocities and accelerations by considering the time step Δt to consist of two equal sub-steps of size $\Delta t/2$. Therefore the method is a 'composite scheme'. For the first sub-step solution, the well-known trapezoidal rule is used, and for the second sub-step solution, the well-known 3-point Euler backward formula is employed.

Hence in the first sub-step, the equations solved are Eq. (1) applied at time $t + \Delta t/2$

$$\mathbf{M}\,{}^{t+\Delta t/2}\ddot{\mathbf{U}} + \mathbf{C}\,{}^{t+\Delta t/2}\dot{\mathbf{U}} = {}^{t+\Delta t/2}\mathbf{R} - {}^{t+\Delta t/2}\mathbf{F} \tag{2}$$

with the equations of the trapezoidal rule

$${}^{t+\Delta t/2}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + \left[\frac{\Delta t}{4}\right]({}^{t}\ddot{\mathbf{U}} + {}^{t+\Delta t/2}\ddot{\mathbf{U}}) \tag{3}$$

$${}^{t+\Delta t/2}\mathbf{U} = {}^{t}\mathbf{U} + \left[\frac{\Delta t}{4}\right]({}^{t}\dot{\mathbf{U}} + {}^{t+\Delta t/2}\dot{\mathbf{U}}) \tag{4}$$

Using Eqs. (2)–(4), the unknown nodal displacements, velocities, and accelerations at time $t + \Delta t/2$ are calculated.

Any iterative scheme can be used, but we employ the Newton–Raphson iteration, see Ref. [1], with the governing equations, for $i = 1, 2, 3, \ldots$

$$\begin{aligned}\left(\frac{16}{\Delta t^2}\mathbf{M}+\frac{4}{\Delta t}\mathbf{C}+{}^{t+\Delta t/2}\mathbf{K}^{(i-1)}\right)\Delta\mathbf{U}^{(i)} &= {}^{t+\Delta t/2}\mathbf{R}-{}^{t+\Delta t/2}\mathbf{F}^{(i-1)}\\ &\quad -\mathbf{M}\left(\frac{16}{\Delta t^2}\left({}^{t+\Delta t/2}\mathbf{U}^{(i-1)}-{}^{t}\mathbf{U}\right)-\frac{8}{\Delta t}{}^{t}\dot{\mathbf{U}}-{}^{t}\ddot{\mathbf{U}}\right)\\ &\quad -\mathbf{C}\left(\frac{4}{\Delta t}\left({}^{t+\Delta t/2}\mathbf{U}^{(i-1)}-{}^{t}\mathbf{U}\right)-{}^{t}\dot{\mathbf{U}}\right)\end{aligned} \quad (5)$$

where

$${}^{t+\Delta t/2}\mathbf{U}^{(i)} = {}^{t+\Delta t/2}\mathbf{U}^{(i-1)} + \Delta\mathbf{U}^{(i)} \quad (6)$$

Once convergence has been reached, we use the calculated displacements ${}^{t+\Delta t/2}\mathbf{U}$ in Eqs. (3) and (4) to obtain the velocities, and accelerations if requested, at time $t + \Delta t/2$.

Then in the second sub-step, the equations solved are Eq. (1) applied at time $t + \Delta t$

$$\mathbf{M}{}^{t+\Delta t}\ddot{\mathbf{U}} + \mathbf{C}{}^{t+\Delta t}\dot{\mathbf{U}} = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F} \quad (7)$$

with the equations of the three-point Euler backward method

$${}^{t+\Delta t}\dot{\mathbf{U}} = \frac{1}{\Delta t}{}^{t}\mathbf{U} - \frac{4}{\Delta t}{}^{t+\Delta t/2}\mathbf{U} + \frac{3}{\Delta t}{}^{t+\Delta t}\mathbf{U} \quad (8)$$

$${}^{t+\Delta t}\ddot{\mathbf{U}} = \frac{1}{\Delta t}{}^{t}\dot{\mathbf{U}} - \frac{4}{\Delta t}{}^{t+\Delta t/2}\dot{\mathbf{U}} + \frac{3}{\Delta t}{}^{t+\Delta t}\dot{\mathbf{U}} \quad (9)$$

Using Eqs. (7)–(9), and the earlier solutions obtained for time $t + \Delta t/2$, the governing equations for the Newton–Raphson iteration to obtain the solution at time $t + \Delta t$ are

$$\begin{aligned}\left(\frac{9}{\Delta t^2}\mathbf{M}+\frac{3}{\Delta t}\mathbf{C}+{}^{t+\Delta t}\mathbf{K}^{(i-1)}\right)\Delta\mathbf{U}^{(i)} &= {}^{t+\Delta t}\mathbf{R}-{}^{t+\Delta t}\mathbf{F}^{(i-1)}\\ &\quad -\mathbf{M}\left(\frac{9}{\Delta t^2}{}^{t+\Delta t}\mathbf{U}^{(i-1)}-\frac{12}{\Delta t^2}{}^{t+\Delta t/2}\mathbf{U}+\frac{3}{\Delta t^2}{}^{t}\mathbf{U}-\frac{4}{\Delta t}{}^{t+\Delta t/2}\dot{\mathbf{U}}+\frac{1}{\Delta t}{}^{t}\dot{\mathbf{U}}\right)\\ &\quad -\mathbf{C}\left(\frac{3}{\Delta t}{}^{t+\Delta t}\mathbf{U}^{(i-1)}-\frac{4}{\Delta t}{}^{t+\Delta t/2}\mathbf{U}+\frac{1}{\Delta t}{}^{t}\mathbf{U}\right)\end{aligned} \quad (10)$$

and

$${}^{t+\Delta t}\mathbf{U}^{(i)} = {}^{t+\Delta t}\mathbf{U}^{(i-1)} + \Delta\mathbf{U}^{(i)} \quad (11)$$

The tangent stiffness matrices ${}^{t+\Delta t/2}\mathbf{K}$ and ${}^{t+\Delta t}\mathbf{K}$ used in Eqs. (5) and (10) are of course the consistent tangent stiffness matrices. Here, large deformations and inelastic response can directly be included [1,15,16]. Also, contact conditions can directly be accounted for in terms of Lagrange multipliers or penalty procedures, always considering only the conditions at the discrete times used above [1,17].

This scheme is a fully implicit second-order accurate method and requires per step about twice the computational effort as the trapezoidal rule. However, the accuracy per time step is good and in particular the method remains stable when the trapezoidal rule fails to give the solution.

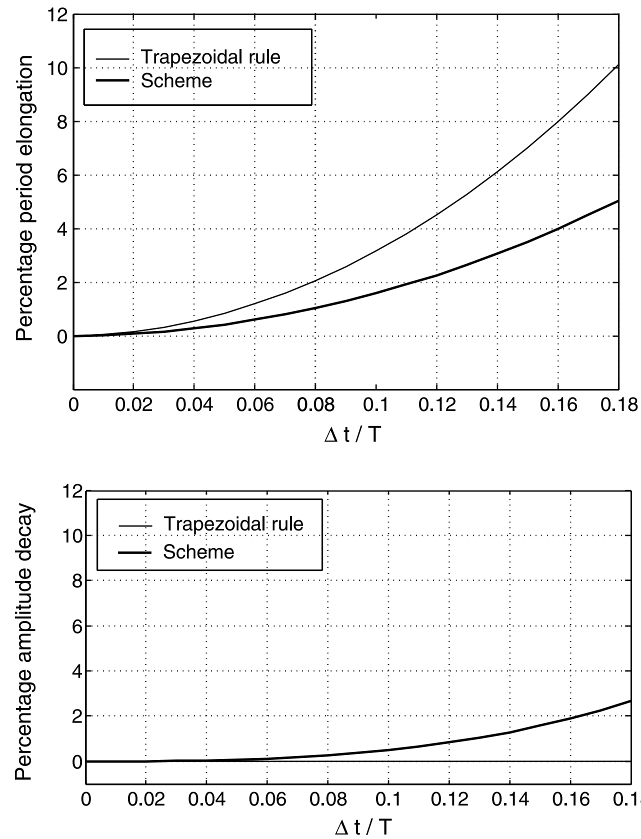

Fig. 1. Percentage period elongation and amplitude decay for the trapezoidal rule and the scheme.

Fig. 1 gives the period elongation and amplitude decay of the scheme, evaluated as in Ref. [1] for a linear system. This behavior was already reported in Ref. [14] but is given here again to be able to refer to it in Section 3. The period elongation is compared with the values of the trapezoidal rule. Of course, the trapezoidal rule does not give an amplitude decay.

3. Solutions of demonstrative problems

In this section we use the scheme to solve two demonstrative problems that we have selected to compare the performance of the scheme with results given earlier using other methods. For this purpose, we consider the pendulum problems solved by Kuhl and Crisfield [3]. Such a truss problem is simple to set up and the analysis gives considerable insight, see also Refs. [1,18]. For the solution of additional illustrative problems, we refer to Refs. [14] and [15].

3.1. Solution of the ‘stiff pendulum problem’

Fig. 2, taken from Ref. [3], describes the pendulum problem. The values used here are close to those given in Ref. [1]. We have modeled the pendulum in ADINA using one two-node truss element with a consistent mass matrix.

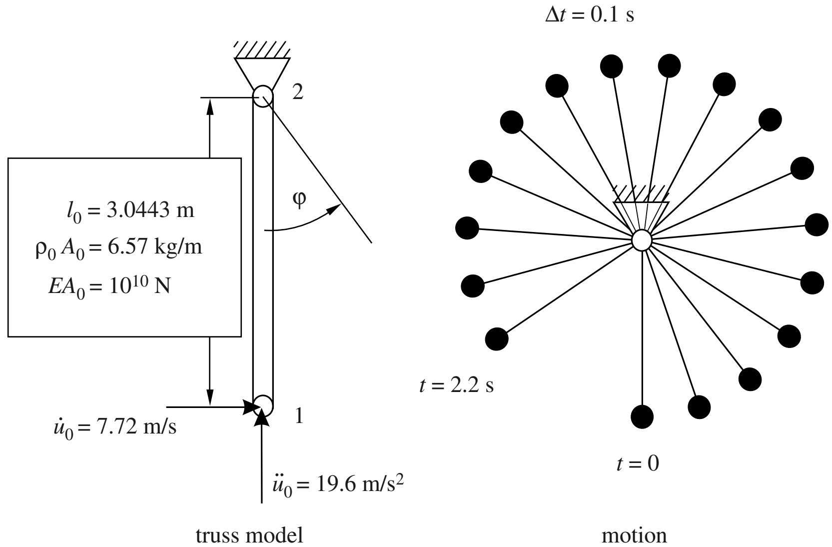

Fig. 2. Simple pendulum with boundary and initial conditions.

The analytical rotational period is 2.4777 s. All solutions are given in the units of Fig. 2.

The solution of the problem for 10 cycles is shown in Fig. 3 using the time step $\Delta t = 0.01$. We also ran the same problem with the same time step for 400 cycles (see Fig. 4) and the solution is stable and shows even after that many cycles hardly an amplitude decay and period elongation, see Fig. 4. In each case, the total amplitude decay is less than 0.0043% and the total period elongation is only about 0.031%. Of course, the period elongation is larger than the amplitude decay as must be expected considering Fig. 1.

If we run the analysis with a time step $\Delta t = 0.6$, we obtain the solution given in Fig. 5. Here one single time step is used for about a 90° rotation of the pendulum (hence one sub-step corresponds to about 45°) and the solution is still stable although of course not accurate. The period elongation is very large with the amplitude decay smaller. In fact, even when using a time step larger than the period, a stable solution was obtained but of course, the response prediction cannot be accurate.

We should note that Ref. [3] already reports that the trapezoidal rule is unstable in this response solution, and Ref. [3] also summarizes that the generalized alpha-method (and related schemes) can be used to solve the problem – however, provided appropriate integration constants for this specific problem are chosen.

3.2. Solution of the 'elastic pendulum problem'

In this next study, we change the stiffness of the truss element like was done in Ref. [3]. The stiffness of the truss element is reduced from $EA_0 = 10^{10}$ N to $EA_0 = 10^4$ N, hence the element is now flexible and shows an axial vibration. Also, following Ref. [3] we now use the initial radial acceleration $\ddot{u}_0 = 0.0$ m/s^2.

Fig. 6 gives the computed response using the time step $\Delta t = 0.01$. The period of axial vibration is 0.28 s and so

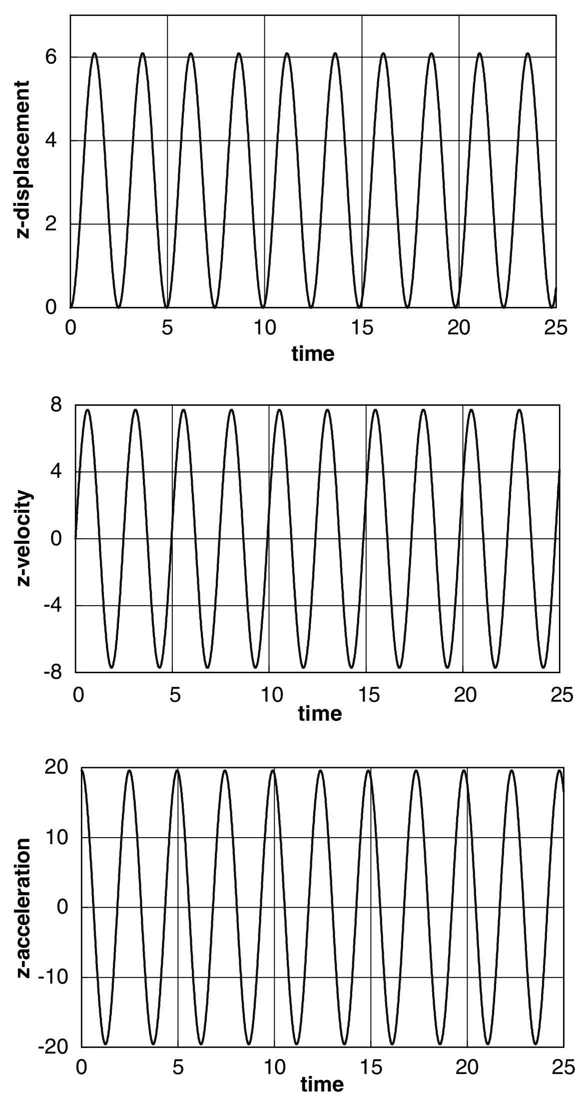

Fig. 3. Stiff pendulum in 10 cycles using the scheme with $\Delta t = 0.01$.

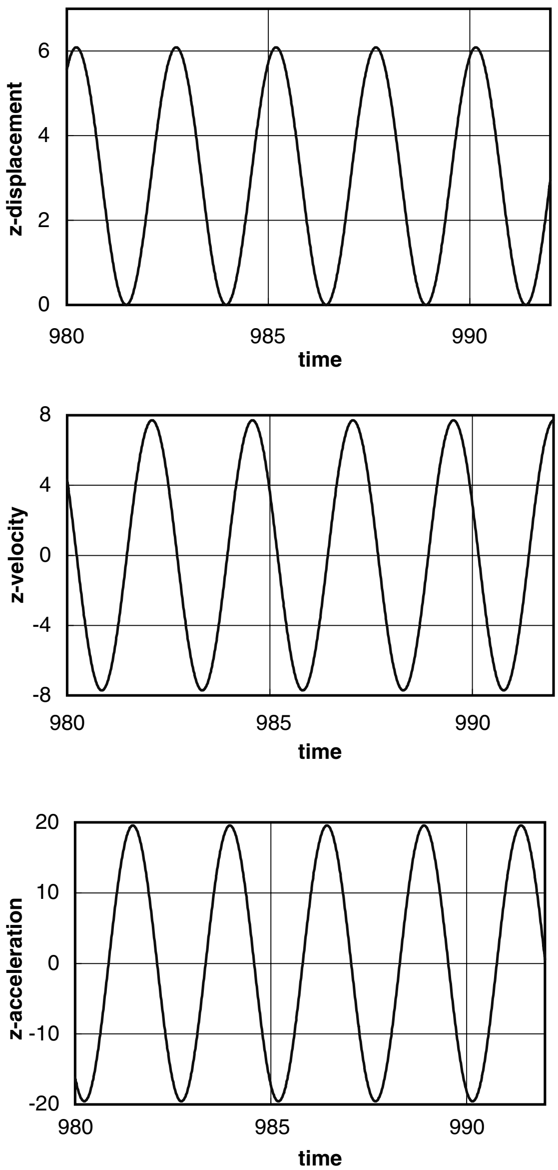

Fig. 4. Stiff pendulum in 400 cycles using the scheme with $\Delta t = 0.01$.

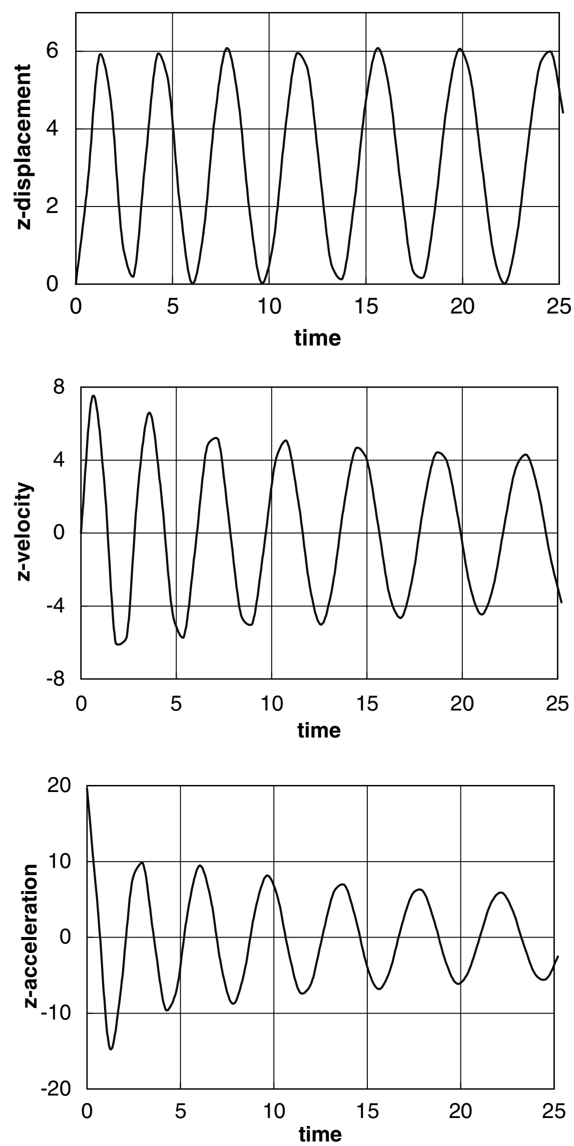

Fig. 5. Stiff pendulum using the scheme with $\Delta t = 0.6$.

the time step employed here corresponds to 28 steps per cycle, which is a reasonable time step size to capture this vibration. The predicted response is quite accurate. This problem can actually also directly be solved with other integration schemes, see Ref. [3]. The solution obtained using the trapezoidal rule is given in Fig. 7.

If a much larger time step is used, namely $\Delta t = 0.05$, giving only about six steps per period of axial vibration, the computed response is stable and overall still quite accurate, but, as must be expected, the axial vibration is predicted quite inaccurately, see Fig. 8. Here the trapezoidal rule yields a more accurate level of strain, see Fig. 9.

4. Discussion of properties of scheme

Considering the solutions given in Section 3 and the ideal properties of an effective time integration scheme summarized in Section 1, we can make a number of observations.

The scheme is very simple with no parameters to adjust, and dynamic equilibrium is established at the discrete times without weighing forces or stresses, all just like in the use of the trapezoidal rule. The method is second-order accurate and of course directly applicable in large deformation analyses including inelastic material behavior and contact conditions. The scheme does not directly impose energy and momentum conservation by means of Lagrange multipliers or a special 'for dynamic analysis' element formulation. This is an important advantage, because the method can thus be employed also for the practical problems in which the mechanical energy is not preserved. This advantage also means that Lagrange multipliers are not solved for, and the same element stiffness matrices as in static analysis are directly used. Also, the coefficient matrices are of course symmetric. With these properties and the accuracy

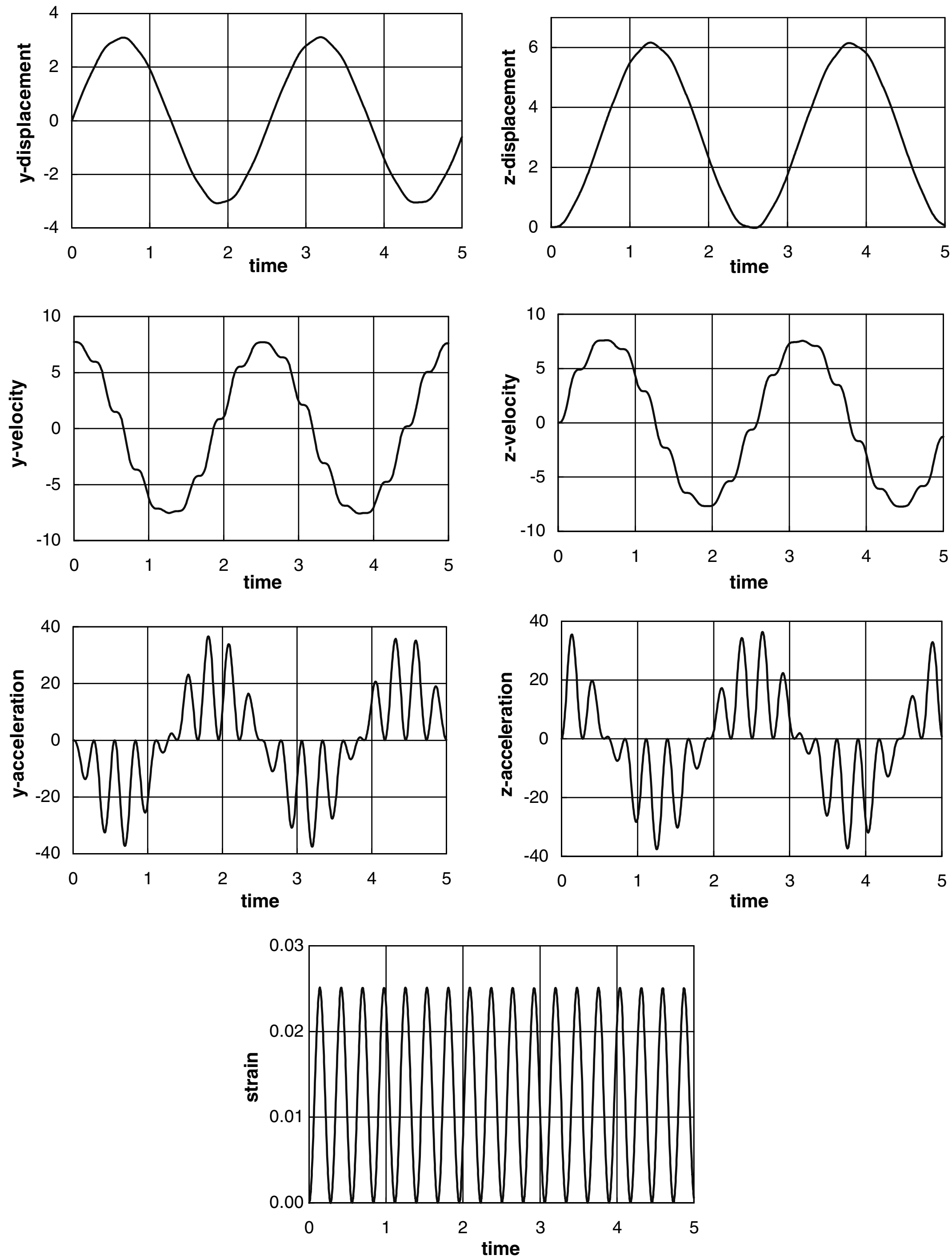

Fig. 6. Elastic pendulum using the scheme with $\Delta t = 0.01$.

obtained in the response solutions, the scheme is quite effective.

The time integration procedure is employed much like all implicit direct time integration schemes are used – in general – in linear and nonlinear analyses. The physical problem is idealized in a mathematical model, which is based on the geometry, material properties, the loading and the displacement boundary conditions. The finite element mesh is chosen to accurately represent all frequencies that are significantly excited by the loading. The time step size is selected to integrate all dynamic response in the frequencies that are significantly excited. The frequencies of the finite element mesh above these significantly excited frequencies are usually inaccurate when compared with the frequencies of the continuum, and any response in these modes of the finite element model will at most correspond to a static contribution, see Ref. [1] for a discussion.

If the scheme presented in this paper is used in this way, satisfactory results can be expected. But, of course, the time integration scheme is not perfect. Ideally, the same proper-

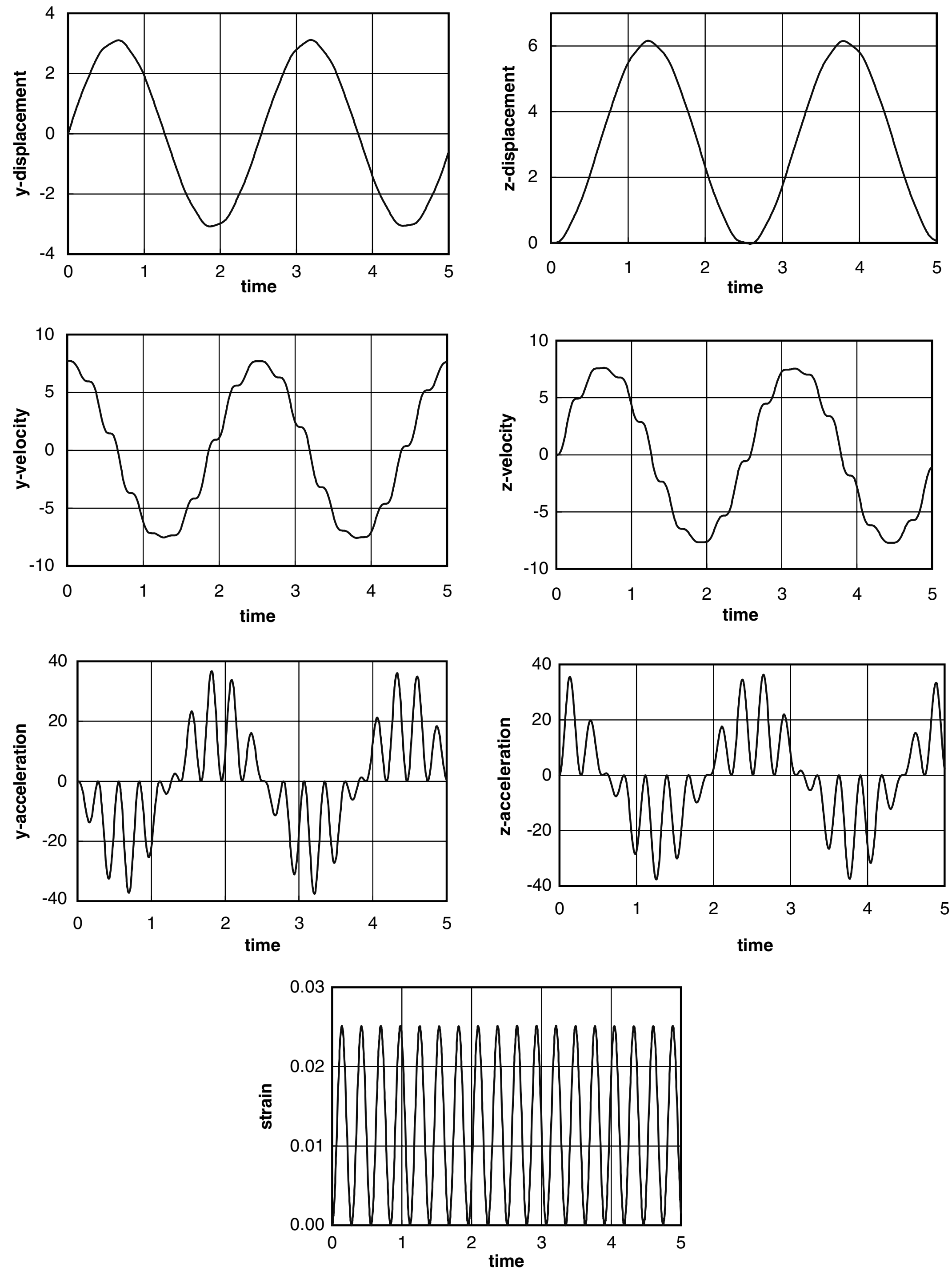

Fig. 7. Elastic pendulum using the trapezoidal rule with $\Delta t = 0.01$.

ties would be present but the computational expense would be less per time step and the accuracy would still be better.

We note that two different coefficient matrices are used in the given time step solution, see Eqs. (5) and (10). This feature does not add difficulty or computational expense in nonlinear analyses based on Newton–Raphson iterations. However, if the scheme is used in linear analysis, the use of a single coefficient matrix is clearly preferable because then only one matrix needs to be factorized prior to the step-by-step solution [1]. Referring to Ref. [14], we use in linear analysis the more general equations given therein with $\gamma = 2 - \sqrt{2}$, which is the value that gives a single coefficient matrix and gives the optimal solution in linear analysis. However, using this value, the error curves in Fig. 1 are hardly affected and for nonlinear analysis, the Eqs. (5) and (10) are simpler. Also, with the scheme presented above, we have had good numerical experiences in large deformation, long time duration analyses.

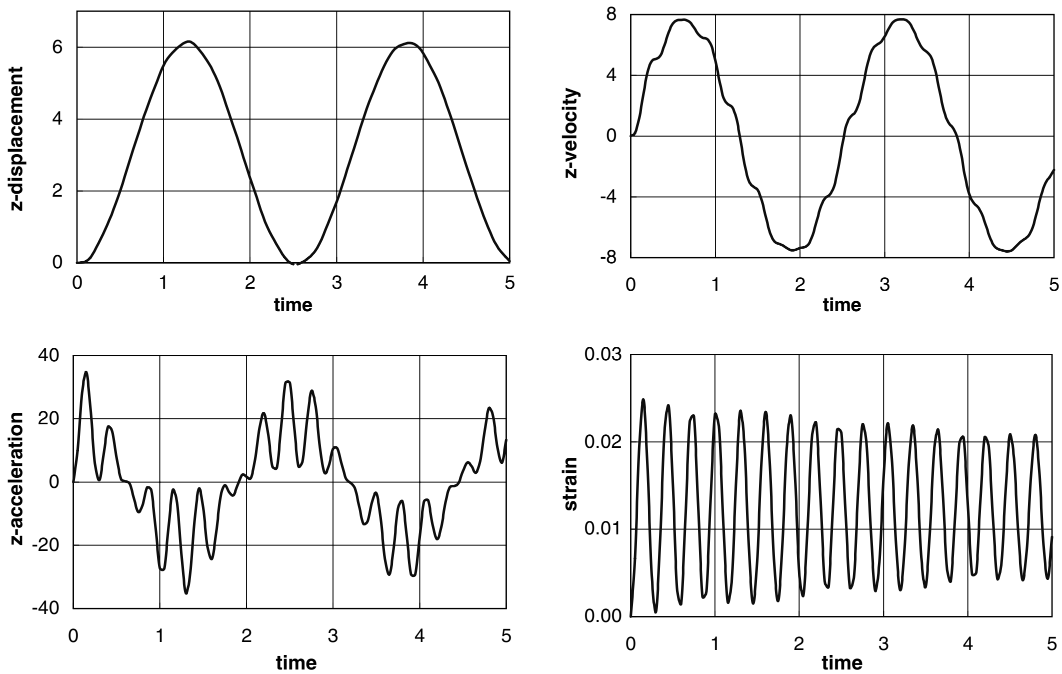

Fig. 8. Elastic pendulum using the scheme with $\Delta t = 0.05$.

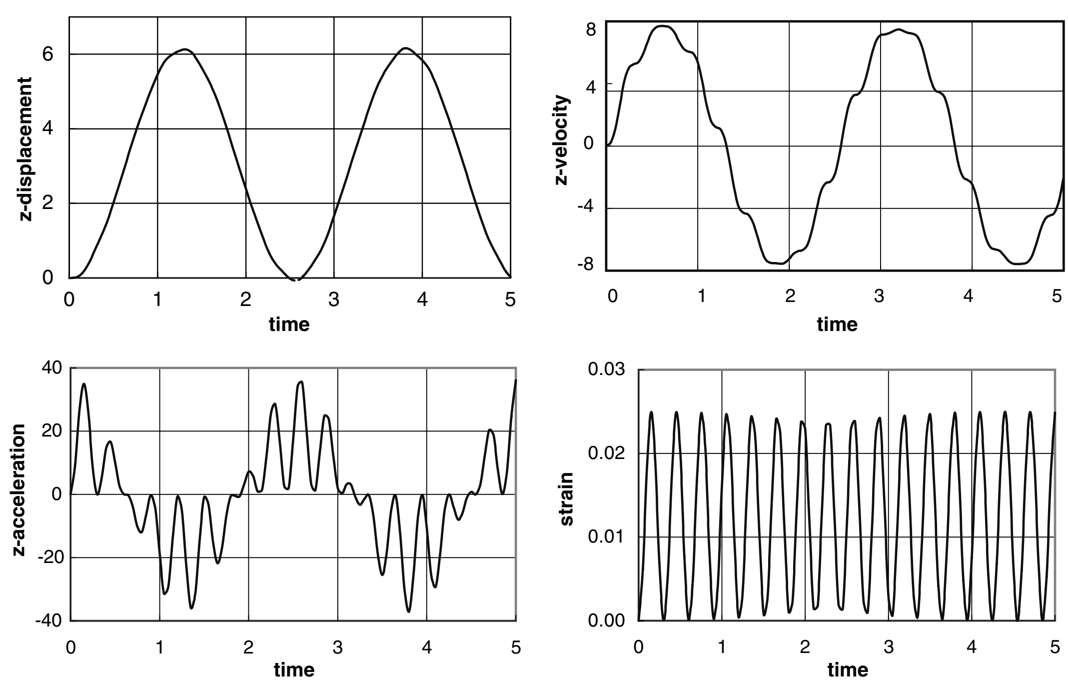

Fig. 9. Elastic pendulum using the trapezoidal rule with $\Delta t = 0.05$.

5. Concluding remarks

The objective in this paper was to focus on a direct implicit time integration scheme that is valuable for some large deformation long time duration dynamic response calculations. The scheme is simple and quite effective, and certainly of use when the trapezoidal rule is not effective and even fails.

The scheme can be viewed as an extension of the trapezoidal rule in that the trapezoidal rule is combined with the 3-point Euler backward method. The procedure is a second-order accurate scheme with small amplitude decay and period elongation. The computations per time step involve only symmetric matrices, and nodal displacement, velocity and acceleration vectors, like when using the trapezoidal rule, but two sub-steps per time step are used.

Hence the method is about twice as expensive as the trapezoidal rule per time step, but less steps can frequently be used and the method remains stable when the trapezoidal rule fails.

The important attribute is that no Lagrange multipliers, or special element formulations for the dynamic analysis are used to conserve energy and momentum. Numerical experiences have shown that these properties are directly satisfied to sufficient accuracy by the time integration scheme. Of course, the accuracy achieved depends on the size of the time step used. The method can therefore also directly be employed when the mechanical energy is not conserved, like in general inelastic and contact analyses.

Naturally, it would be valuable to pursue mathematical analyses of the scheme that prove the characteristics observed numerically. These analyses will also be valuable to possibly improve the scheme, and to establish a posteriori error measures for an obtained solution and for automatic time stepping [19,20]. Such mathematical analyses are left for further studies of the scheme.

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