On direct time integration in large deformation dynamic analysis

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Abstract

Direct time integration is used widely for the solution of large deformation problems in solid and structural mechanics. Direct integration schemes that are unconditionally stable for linear dynamic problems are also used for nonlinear problems. However, unconditional stability may be lost in the nonlinear regime. The Newmark method (trapezoidal rule) is used widely but may become unstable when large deformations and long time durations are considered. A composite scheme is proposed for such analysis cases, and the results obtained using the trapezoidal rule and the composite formula in a test problem are given. These results indicate the value of the composite scheme.

Keywords: Direct time integration; Nonlinear dynamic analysis; Stability; Large displacements

1. Introduction

Direct time integration is used widely for the analysis of nonlinear dynamic problems in structural mechanics. A procedure is applied to solve the following equations of equilibrium

$$\mathbf{M}\ddot{\mathbf{U}} = \mathbf{R} - \mathbf{F} \tag{1}$$

where for simplicity we are not considering velocity-dependent damping.

Either explicit or implicit integration can be used, depending on the nature of the problem to be solved. In explicit integration, Eq. (1) is satisfied at time t; based on that, the solution at time $t + \Delta t$ is obtained. The solution marches through time without ever having to solve a system of equations, with relatively little computational cost per time step. Explicit schemes are conditionally stable, though, and may need a time-step size that is much smaller than that demanded by accuracy. These methods are therefore better suited to problems such as wave propagation and crash conditions, in which higher modes are excited and a fine resolution of the solution is required.

Implicit methods require the solution of a system of equations at each time step. Use of unconditionally stable implicit schemes in linear analysis enables the use of relatively large time steps, the size of which is dictated only by accuracy considerations [1]. However, in nonlinear analyses, such schemes can cease to be stable, and for time-step sizes that should be small enough for the required accuracy, the solution may blow up. This is exhibited by a lack of energy and momentum conservation in the solution for conservative problems.

Much research has been expended to overcome this lack of stability in the nonlinear regime. Energy momentum methods are shown by Simo and Tarnow [2] to numerically conserve the energy and angular momenta exactly for conservative problems. These algorithms have been shown to remain stable for long time simulations, where the more traditional algorithms such as the trapezoidal rule have failed to produce a solution. However, as shown by Laursen and Meng [3], for general nonlinear material models these methods require solution of a scalar variable either at the integration points or over the element in an averaged sense. Also, a nonsymmetric tangent stiffness matrix may be obtained, all making the algorithms computationally considerably more expensive.

We are proposing in this paper an algorithm that operates only on global vectors, uses only the usual symmetric matrices, and retains good stability and accuracy characteristics.

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2. A composite trapezoidal-backward-difference procedure

In general, backward-difference expressions, when used to approximate time derivatives in the equation of dynamic equilibrium, result in algorithms that are dissipative, i.e. a certain amount of numerical damping is introduced in the solution. The Houbolt method is one such example that uses a four-point backward-difference approximation but introduces too much error even in lower modes [1].

A composite integration scheme for solving first-order equations arising in the simulation of silicon devices and circuits was presented by Bank et al. [4]. This composite scheme is being used in the ADINA program for fluid-flow structural interaction problems. The first-order fluid-flow equations and second-order structural equations are solved fully coupled in time using this procedure [5,6]. In this paper, we present the time integration scheme for the equations of solids and structures, i.e. Eq. (1), and give the solution of a test problem to compare the scheme with the usual trapezoidal rule integration. For details on the notation used, see Bathe [1].

Assume that the solution is known completely at time t; based on that, the solution at time $t + \Delta t$ is to be computed. Let $t + \gamma\Delta t$ be an instant in time between times t and $t + \Delta t$, i.e. $\gamma \in (0, 1)$. Then, using the trapezoidal rule over the time interval $\gamma\Delta t$, we have the following assumptions on velocity and displacement:

$$^{t+\gamma\Delta t}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + \frac{^{t}\ddot{\mathbf{U}} + {}^{t+\gamma\Delta t}\ddot{\mathbf{U}}}{2}\gamma\Delta t \quad (2)$$

and

$$^{t+\gamma\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + \frac{^{t}\dot{\mathbf{U}} + {}^{t+\gamma\Delta t}\dot{\mathbf{U}}}{2}\gamma\Delta t \quad (3)$$

or after simplification,

$$^{t+\gamma\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + {}^{t}\dot{\mathbf{U}}\gamma\Delta t + ({}^{t}\ddot{\mathbf{U}} + {}^{t+\gamma\Delta t}\ddot{\mathbf{U}})\left(\frac{\gamma\Delta t}{2}\right)^2 \quad (4)$$

Solving for $^{t+\gamma\Delta t}\ddot{\mathbf{U}}$

$$^{t+\gamma\Delta t}\ddot{\mathbf{U}} = ({}^{t+\gamma\Delta t}\mathbf{U} - {}^{t}\mathbf{U} - {}^{t}\dot{\mathbf{U}}\gamma\Delta t)\frac{4}{\gamma^2\Delta t^2} - {}^{t}\ddot{\mathbf{U}} \quad (5)$$

The equilibrium equation, Eq. (1), at time $t + \gamma\Delta t$ can be written as

$$\mathbf{M}^{t+\gamma\Delta t}\ddot{\mathbf{U}} = {}^{t+\gamma\Delta t}\mathbf{R} - {}^{t+\gamma\Delta t}\mathbf{F} \quad (6)$$

Substituting for $^{t+\gamma\Delta t}\ddot{\mathbf{U}}$ in the above equation, and linearizing the equation about the most recent configuration, the following expression is obtained [1]:

$$\begin{aligned}\left({}^{t+\gamma\Delta t}\mathbf{K}^{(i-1)} + \mathbf{M}\frac{4}{\gamma^2\Delta t^2}\right)\Delta\mathbf{U}^{(i)} &= {}^{t+\gamma\Delta t}\mathbf{R} - {}^{t+\gamma\Delta t}\mathbf{F}^{(i-1)} \\ &-\mathbf{M}\left(\frac{4}{\gamma^2\Delta t^2}({}^{t+\gamma\Delta t}\mathbf{U}^{(i-1)} - {}^{t}\mathbf{U}) \right. \\ &\left. -\frac{4}{\gamma\Delta t}{}^{t}\dot{\mathbf{U}} - {}^{t}\ddot{\mathbf{U}}\right)\end{aligned} \quad (7)$$

Once the displacements have been computed, the velocities and accelerations are obtained from the relations given above.

Let the derivative of a function at time $t + \Delta t$ be written in terms of the function values at times t, $t + \gamma\Delta t$, and $t + \Delta t$ as

$$^{t+\Delta t}\dot{f} = c_1{}^{t}f + c_2{}^{t+\gamma\Delta t}f + c_3{}^{t+\Delta t}f \quad (8)$$

where

$$c_1 = \frac{(1-\gamma)}{\Delta t\gamma} \quad (9)$$

$$c_2 = \frac{-1}{(1-\gamma)\gamma\Delta t} \quad (10)$$

$$c_3 = \frac{(2-\gamma)}{(1-\gamma)\Delta t} \quad (11)$$

Therefore, writing velocities in terms of displacements and accelerations in terms of velocities, we have

$$^{t+\Delta t}\dot{\mathbf{U}} = c_1{}^{t}\mathbf{U} + c_2{}^{t+\gamma\Delta t}\mathbf{U} + c_3{}^{t+\Delta t}\mathbf{U} \quad (12)$$

$$^{t+\Delta t}\ddot{\mathbf{U}} = c_1{}^{t}\dot{\mathbf{U}} + c_2{}^{t+\gamma\Delta t}\dot{\mathbf{U}} + c_3{}^{t+\Delta t}\dot{\mathbf{U}} \quad (13)$$

Now, writing Eq. (1) at time $t + \Delta t$ and making use of the above expressions gives

$$\mathbf{M}^{t+\Delta t}\ddot{\mathbf{U}} = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F} \quad (14)$$

which, on linearizing, results in

$$\begin{aligned}(c_3c_3\mathbf{M} + {}^{t+\Delta t}\mathbf{K}^{(i-1)})\Delta\mathbf{U}^{(i)} &= {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{(i-1)} \\ &-\mathbf{M}(c_1{}^{t}\dot{\mathbf{U}} + c_2{}^{t+\gamma\Delta t}\dot{\mathbf{U}} + c_3c_1{}^{t}\mathbf{U} \\ &+c_3c_2{}^{t+\gamma\Delta t}\mathbf{U} + c_3c_3{}^{t+\Delta t}\mathbf{U}^{(i-1)})\end{aligned} \quad (15)$$

The solution for $^{t+\Delta t}\mathbf{U}$ and the calculation of the velocities and accelerations from the backward difference approximations in Eqs (12) and (13) gives the complete response at time $t + \Delta t$.

3. Numerical example

To test the stability of the proposed algorithm we choose a problem in elastodynamics, very similar to the one used by Laursen and Meng [3]. A plate in plane stress conditions is subjected to loading, as shown in Fig. 1. The load is applied only for 10 s to give the plate

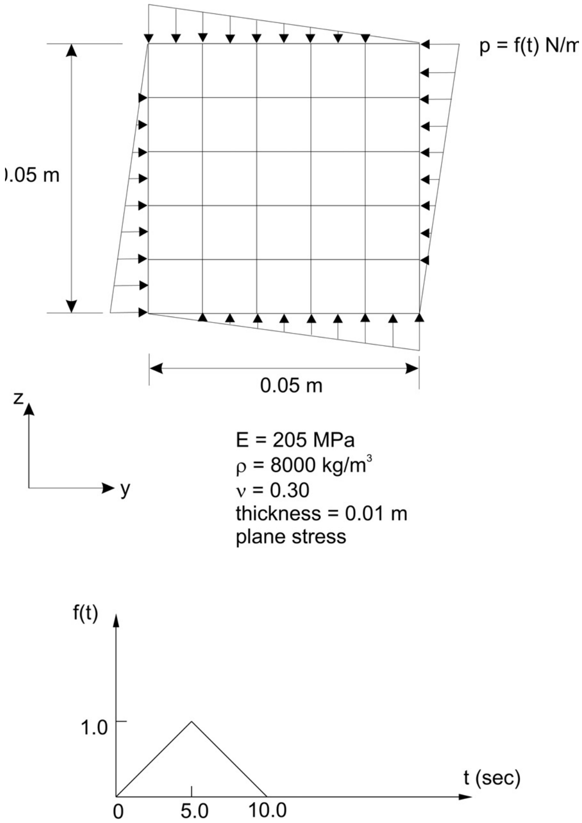

Fig. 1. The rotating plate problem.

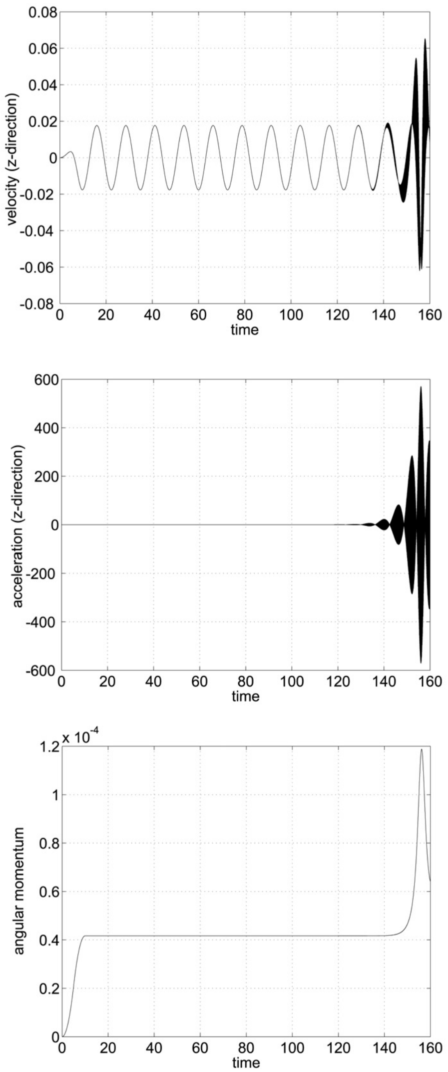

Fig. 2. The rotating plate problem. Results using the trapezoidal rule; $\Delta t = 0.02\,\mathrm{s}$.

a reasonable angular velocity and is then taken off to produce a conservative system from that instant onwards.

The problem is first solved using the trapezoidal rule with $\Delta t = 0.02\,\mathrm{s}$. The velocity and acceleration in the z-direction of the initially top right corner of the plate are plotted along with the angular momentum in Fig. 2. The response of the structure under the specified loading is contained mainly within the rigid body rotational mode. The period of rigid body rotation is about 12.5 s and, therefore, the time step chosen should be small enough to capture the response very accurately. Even with such a small time step, the trapezoidal rule fails to produce the correct solution after a number of revolutions. In fact, the errors begin to show up in the acceleration response, which of course contributes to the deterioration in the quality of the velocity response, resulting in an eventual blow-up in the displacements. Consequently, the angular momentum is not conserved as well, and a point is reached at which the solution can not proceed any further.

The same problem is next solved using the proposed composite formula with $\gamma = 0.5$ and $\Delta t = 0.4\,\mathrm{s}$. It is seen in Fig. 3 that the quality of response remains excellent. Actually, there is a negligible decay in the angular momentum of the plate. This decay is less than 0.06% per revolution for the time step chosen. This problem solution illustrates the superior and more robust performance of the composite procedure because the solution remains stable and accurate for a very long duration.

4. Conclusions

We have presented a composite time integration scheme based on a combination of the trapezoidal rule

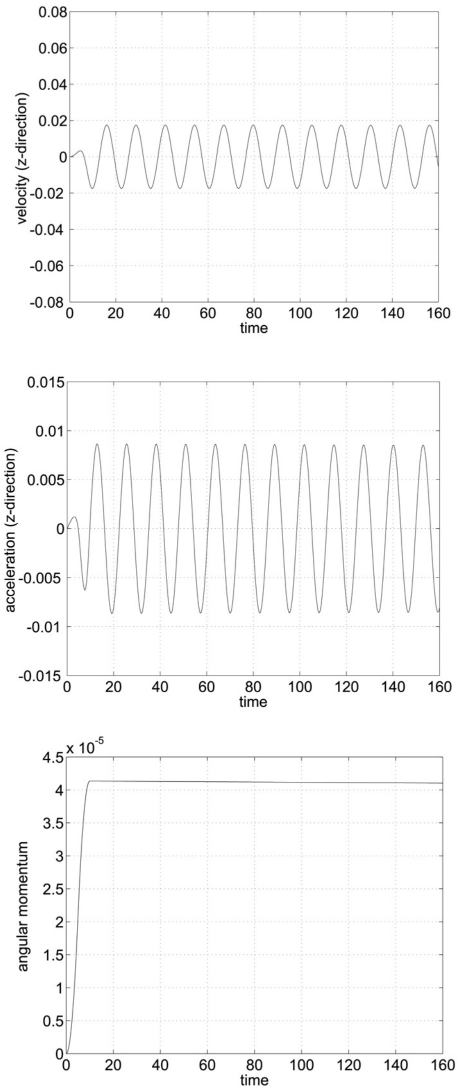

Fig. 3. The rotating plate problem. Results using the composite scheme; $\Delta t = 0.4\,\mathrm{s}$.

and a three-point backward-difference approximation. The composite scheme is used in the ADINA program for the solution of fluid-flow structural interaction problems, where first- and second-order system equations are fully coupled. The method is typically used in ADINA for the analysis of fluid-flow-induced vibrations of structures.

In this paper, we presented the scheme for structural analyses and demonstrated the performance in a test problem. For a given time step size, the scheme is about twice as expensive computationally as the usual trapezoidal rule and, hence, the method is of interest only for analysis cases where the composite scheme provides much more stability and accuracy than the trapezoidal rule. A typical case was presented in this paper and involves large displacements and rotations over long periods of time. Further details and experiences with the algorithm are given in Baig and Bathe [7].

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