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ON A THEORETICAL AND COMPUTATIONAL FRAMEWORK FOR LARGE STRAIN ANISOTROPIC PLASTICITY

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Summary. *We consider the modelling of large strain anisotropic plasticity based on hyperelasticity, the multiplicative decomposition into the elastic and plastic deformation gradients, and the use of the logarithmic strain measure and its work-conjugate stress measure. The integration of the plastic deformation gradient is performed using an exponential mapping. First, we give the continuum mechanics formulation of our proposed theory, and then we summarize some results considering anisotropic elastic behavior, anisotropic yield functions and mixed hardening.*

1 INTRODUCTION

The development of effective large strain elasto-plasticity formulations has been given much attention during the recent years, see for example [1, 2] and the references therein. While isotropic behavior can be modelled quite effectively, the effective solution of anisotropic behavior involving anisotropic hyperelasticity combined with anisotropic yield functions and mixed hardening is much more difficult, theoretically and computationally. In these cases, the plastic spin may need to be modelled and this requires special considerations. The objective of this presentation is to give a framework for modelling large strain plasticity with these anisotropic effects.

2 THE CONTINUUM FORMULATION

We use the notation of refs. [3–6] because it is a natural notation to use when actual solution algorithms are proposed. Let ${}_0^t\mathbf{X}$ be the deformation gradient at time t. We base our theory on the Lee decomposition, so the deformation gradient is decomposed into an elastic part, $\mathbf{X}^e$, and a plastic part, $\mathbf{X}^p$, with $\mathbf{X} = \mathbf{X}^e\mathbf{X}^p$. The spatial velocity gradient $\mathbf{l}$ is decomposed additively into an elastic and a plastic part, where the plastic part is

$\mathbf{l}^p = \mathbf{X}^e \left[\dot{\mathbf{X}}^p \left(\mathbf{X}^p \right)^{-1} \right] \left(\mathbf{X}^e \right)^{-1}$ and $\mathbf{L}^p := \dot{\mathbf{X}}^p \left(\mathbf{X}^p \right)^{-1}$ is the modified plastic velocity gradient. The symmetric part of $\mathbf{L}^p$ is the modified plastic deformation tensor, $\mathbf{D}^p = sym\left(\mathbf{L}^p\right)$, whereas the skew part is the modified plastic spin, $\mathbf{W}^p = skw\left(\mathbf{L}^p\right)$. The logarithmic strain tensor is defined as $\mathbf{E}^e := \frac{1}{2} \ln \mathbf{C}^e$, where $\mathbf{C}^e := \mathbf{X}^{e\ T} \mathbf{X}^e$. Obviously, different strain measures and their work-conjugate stress measures may always be related by fourth order mapping tensors. In particular the symmetric part, $\boldsymbol{\Xi}_s$, and the skew part, $\boldsymbol{\Xi}_w$, of the Mandel stress tensor $\boldsymbol{\Xi} = \mathbf{C}^e \mathbf{S}$ (where $\mathbf{S}$ is the second Piola-Kirchhoff stress tensor) may be related to a generalized Kirchhoff stress tensor $\mathbf{T}$, work conjugate to the logarithmic strains, by (see [7, 8])

$$\boldsymbol{\Xi}_s = \mathbf{T} : \mathbb{S}^M \quad \text{and} \quad \boldsymbol{\Xi}_w = \mathbf{T} : \mathbb{W}^M = \mathbf{E}^e\ \mathbf{T} - \mathbf{T}\mathbf{E}^e \tag{1}$$

where $\mathbb{S}^M$ and $\mathbb{W}^M$ are fourth order mapping tensors, functions of the elastic strains (see details in Reference [8]). In the case of isotropic elasticity the tensor $\mathbf{T}$ coincides with the usual Kirchhoff stress tensor $\boldsymbol{\tau}$. The anisotropic stored energy function for the anisotropic material may be assumed to be of the type

$${}^t\mathcal{W} = U\left({}^tJ \right) + \mu\ {}^t\mathbf{E}^e :\ {}^t\mathbb{A} :\ {}^t\mathbf{E}^e \tag{2}$$

where ${}^t\mathbb{A}$ is the elastic anisotropy tensor, whose preferred directions may rotate at a speed given by $\mathbf{W}^A$. In this expression $U\left({}^tJ \right)$ is the volumetric component, $J = \det\left(\mathbf{X}^e\right)$ and μ plays the role of a shear modulus. The tensor ${}^{t+\Delta t}_{\ \ t}\mathbf{R}^w := \exp\left({}^{t+\Delta t}\mathbf{W}^p \Delta t \right)$ may be interpreted as a measure of the incremental plastic rotation due to lattice dislocations (see References [5, 8]). The tensor ${}^{t+\Delta t}_{\ \ t}\mathbf{R}^w$ defines, from the stress-free configuration, a configuration in which the plastic rotations are frozen during plastic flow. We label objects in this configuration by an underlining arrow, i.e. ${}^{t+\Delta t}\underleftarrow{\mathbf{E}}^e = {}^{t+\Delta t}_{\ \ t}\mathbf{R}^{wT}\ {}^{t+\Delta t}\mathbf{E}^e\ {}^{t+\Delta t}_{\ \ t}\mathbf{R}^w$. Using these expressions and definitions, the rate of stored energy function $\dot{\mathcal{W}}$ may be written as

$$\dot{\mathcal{W}} = \mathbf{T} :\ \underleftarrow{\mathcal{L}}\mathbf{E}^e + \mathbf{T}_w : \mathbf{W}^A \tag{3}$$

where $\underleftarrow{\mathcal{L}}(\cdot)$ is a Lie derivative with ${}^{t+\Delta t}_{\ \ t}\mathbf{R}^w$ acting as gradient and $\mathbf{T}_w := \mathbf{E}^e\ \mathbf{T} - \mathbf{T}\ \mathbf{E}^e \equiv \boldsymbol{\Xi}_w$. Similar expressions apply for the hardening function, $\mathcal{H}$, and the rate of hardening, $\dot{\mathcal{H}}$. All these equations, inserted in the dissipation equation yield the following plastic dissipation function

$$\dot{\mathcal{D}}^p := \boldsymbol{\Xi}_s \!:\! \mathbf{D}^p + \boldsymbol{\Xi}_w \!:\! \mathbf{W}^p - \mathbf{T}_w : \mathbf{W}^A - \mathbf{B}_s : \underleftarrow{\mathcal{L}}\mathbf{E}^i - \mathbf{B}_w : \mathbf{W}^H - \kappa\dot{\zeta} - \kappa_w\dot{\xi} \geq 0 \tag{4}$$

where $\mathbf{B}_s$ is the backstress tensor, $\mathbf{B}_w := \mathbf{E}^i\ \mathbf{B}_s - \mathbf{B}_s\ \mathbf{E}^i$ and $\mathbf{E}^i$ are tensorial logarithmic strain-like internal variables. By $\mathbf{W}^H$ we denote the spin of the hardening anisotropy tensor. The underlining arrow in $\underleftarrow{\mathcal{L}}\mathbf{E}^i$ implies a Lie derivative with the internal variables rotations, ${}^{t+\Delta t}_{\ \ t}\mathbf{R}^{wi} := \exp\left({}^{t+\Delta t}\mathbf{W}^i \Delta t \right)$, acting as gradient, where $\mathbf{W}^i$ is the internal variables spin tensor. The scalars κ and κ_w are the effective stress-like internal variables

(current yield stress and yield couple-stress). The scalars ζ and ξ are the effective strain-like internal variables (effective plastic strain and effective plastic rotation).
Assume that the elastic region is enclosed by two yield functions $f_s\left(\boldsymbol{\Xi}_s, \mathbf{B}_s,\ \kappa\right)$ and $f_w\left(\boldsymbol{\Xi}_w, \mathbf{B}_w,\ \kappa_w\right)$. Then, the Lagrangian for the constrained problem is $L = \dot{\mathcal{D}}^p - \dot{t} f_s - \dot{\gamma} f_w$, where $\dot{t}$ and $\dot{\gamma}$ are the consistency parameters. If we claim that the principle of maximum dissipation holds, the stress and other internal variables are such that $\nabla L = 0$, i.e. for the yield function expressions given

$$\nabla L = 0 \Rightarrow \begin{cases} \dfrac{\partial L}{\partial \boldsymbol{\Xi}_s} = 0 \ \Rightarrow\ \mathbf{D}^p = \dot{t}\dfrac{\partial f_s}{\partial \boldsymbol{\Xi}_s} \ \text{ and } \ \dfrac{\partial L}{\partial \boldsymbol{\Xi}_w} = 0 \ \Rightarrow\ \mathbf{W}^d = \dot{\gamma}\dfrac{\partial f_w}{\partial \boldsymbol{\Xi}_w} \\[2ex] \dfrac{\partial L}{\partial \mathbf{B}_s} = 0 \ \Rightarrow \underleftarrow{\mathcal{L}}\mathbf{E}^i = -\dot{t}\dfrac{\partial f_s}{\partial \mathbf{B}_s} \ \text{ and } \ \dfrac{\partial L}{\partial \mathbf{B}_w} = 0 \ \Rightarrow\ \mathbf{W}^H = -\dot{\gamma}\dfrac{\partial f_w}{\partial \mathbf{B}_w} \\[2ex] \dfrac{\partial L}{\partial \kappa} = 0 \ \Rightarrow\ \dot{\zeta} = -\dot{t}\dfrac{\partial f_s}{\partial \kappa} \ \text{ and } \ \dfrac{\partial L}{\partial \kappa_w} = 0 \ \Rightarrow \dot{\xi} = -\dot{\gamma}\dfrac{\partial f_w}{\partial \kappa_w} \end{cases} \tag{5}$$

where $\mathbf{W}^d := \mathbf{W}^p - \mathbf{W}^A$.

3 EVOLUTION OF INTERNAL VARIABLES

The evolution of the internal variables requires insight based on theory and experiment and was a key aspect of our study, which we of course still continue. If, as usual for f_s and then natural for f_w, the enclosure of the elastic region is expressed in the form of $f_s\left(\boldsymbol{\Xi}_s - \mathbf{B}_s, ...\right)$, $f_w\left(\boldsymbol{\Xi}_w - \mathbf{B}_w, ...\right)$, then for associated plasticity $\underleftarrow{\mathcal{L}}\mathbf{E}^i = \mathbf{D}^p$ and $\mathbf{W}^H = \mathbf{W}^d$ are automatically enforced. For the specific logarithmic-like internal strain variables used, the hardening function in terms of those variables and the mentioned form of the yield functions, we obtain the following computational update

$${}^{t+\Delta t}_{\ \ \ 0}\mathbf{E}^i = {}^{t+\Delta t}_{\ \ \ t}\mathbf{R}^{wi}\ {}^t_0\mathbf{E}^i\ {}^{t+\Delta t}_{\ \ \ t}\mathbf{R}^{wi\ T} + \Delta t\ {}^{t+\Delta t}\mathbf{D}^p \tag{6}$$

Since ${}^{t+\Delta t}\mathbf{W}^i$ does not appear in the dissipation equation, it is not specified directly by the maximum dissipation principle. A possible choice is ${}^{t+\Delta t}\mathbf{W}^i = {}^{t+\Delta t}\mathbf{W}^p$, so the modified plastic spin of the stress-free configuration coincides with the spin of the internal variables.

4 OBSERVATIONS

The above theory can of course be used with the assumption that the plastic spin is zero. This assumption is reasonable, and has been used extensively, when von Mises plasticity with isotropic elastic behavior and mixed hardening is considered. However, it is then important to not inadvertently introduce an unknown and uncontrolled plastic spin merely because of assumptions in the numerical integration algorithms. Such cases were analyzed and discussed in ref. [5].

However, the major use of the above theory is in modelling anisotropic behavior which arises due to anisotropic elasticity, anisotropic yield criteria and mixed hardening. The theory takes into account the rotation of the axes of orthotropy which is assumed to be a function of the plastic spin. More details of the theoretical framework and the algorithm used are given in refs [8, 9].

REFERENCES

[1] K.J. Bathe, ed. *Computational Fluid and Solid Mechanics 2001, 2003 and 2005*. Proceedings of the three M.I.T. Conferences on Computational Fluid and Solid Mechanics. Elsevier 2001, 2003, 2005.

[2] E. Oñate, D.R.J. Owen eds. *VII International Conference on Computational Plasticity*. Proceedings. Barcelona 2003.

[3] K.J. Bathe. *Finite Element Procedures*. Prentice-Hall 1996.

[4] M. Kojić, K.J. Bathe. *Inelastic Analysis of Solids and Structures*. Springer 2005.

[5] F.J. Montáns, K.J. Bathe. Computational issues in large strain elasto-plasticity: An algorithm for mixed hardening and plastic spin. *International Journal for Numerical Methods in Engineering*, **63**, 159-196, 2005.

[6] E.N. Dvorkin, M.B. Goldschmit. *Nonlinear Continua*. Springer, in press.

[7] F.J. Montáns, K.J. Bathe. Large strain anisotropic plasticity including effects of plastic spin. *In Computational Fluid and Solid Mechanics 2005*, K.J. Bathe, ed. Elsevier 2005.

[8] F.J. Montáns, K.J. Bathe. A framework for computational large strain plasticity — anisotropic elasticity, anisotropic yield functions and mixed hardening, in preparation.

[9] F.J. Montáns, K.J. Bathe. A computational procedure for large strain plasticity — anisotropic elasticity, anisotropic yield functions and mixed hardening, in preparation.