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# On the ellipticity condition for model-parameter dependent mixed formulations

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ABSTRACT

When establishing and analyzing model-parameter dependent mixed formulations, it is common to consider required ellipticity and inf–sup conditions for the continuous and discrete problems. However, in the modeling of some important categories of problems, like in the analysis of plates and shells, the ellipticity condition usually considered does not naturally hold, and the inf–sup condition can only be stated in an abstract form and can hardly be evaluated analytically. In this paper we present a new and practical ellipticity condition which together with the inf–sup condition guarantees that (i) when the model parameter goes to zero, the limit problem solution is uniformly approached, and (ii) an optimal finite element discretization has been established (for the interpolations used). In practice, a numerical test might be performed to see whether the proposed ellipticity condition is satisfied.



## 1. Introduction

For the modeling and numerical solution of various problems in engineering and the sciences, like in the analysis of (almost) incompressible solids, fluids, thin structures, electro-magnetic effects, and multiphysics phenomena, it is most effective to use mixed formulations. In these analyses, it is natural and frequently necessary to use two (or even more) fields to formulate the governing equations. Indeed, optimal finite element discretizations can mostly only be obtained if mixed formulations are used [1–6].

In this paper we consider two-field mixed problems in which a small physical parameter is present, like frequently encountered in the analysis of (almost) incompressible solids and fluids, and in the analysis of beams, plates and shells, see [1–4]. The conditions to be satisfied in the modeling of the continuous and discrete problems are the associated ellipticity and inf–sup conditions. However, for important categories of problems, and notably the analysis of plates and shells, the ellipticity condition that is usually considered does not naturally hold. Furthermore, the inf–sup condition can only be stated in abstract form and can hardly be evaluated analytically. Numerical tests have therefore been proposed [7–9].

In the cases when the ellipticity condition does not naturally hold, and also in other cases, the approach is frequently to use a stabilization technique, and concentrate on the inf–sup condition to be satisfied. However, in practice, it is important that the relevant ellipticity condition not be violated without stabilization constants.

The importance of satisfying the ellipticity condition is illustrated in Ref. [2, page 474] with an example. The lowest frequencies of a cantilever bracket that are calculated when the ellipticity condition is not satisfied do not include spurious zero frequencies but – actually worse – *non-zero* spurious (or ghost) frequencies. In a dynamic step-by-step solution the mode shapes corresponding to these non-zero frequencies would capture energy in a non-physical manner. This kind of error is difficult to identify in an actual analysis.

These ghost frequencies may in general also be present when stabilization techniques are used, but then usually correspond to higher frequencies. Clearly, such frequencies and mode shapes should best not be present in the finite element model. The key therefore is to satisfy the ellipticity condition directly in a natural manner and without any stabilization technique – like in the displacement-based formulations (using full numerical integration [2]).

Of course, before formulating a finite element discretization, it is necessary to establish the formulation of the continuous problem with the appropriate spaces, ellipticity and inf–sup conditions. In the case of model-parameter dependent problems, it is important to identify the "best" ellipticity condition to practically work with and show that, when the model parameter approaches zero, the solutions converge to the solutions of the limit problem (the model parameter is zero). Thereafter, we can discretize the continuous problem with appropriate finite element spaces and identify the appropriate and corresponding ellipticity condition, which together with the relevant inf–sup condition assures that the finite element discretization is optimal.

While very important results regarding ellipticity and inf–sup conditions have been published, the results so far given – see

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specifically Ref. [3] – are not giving the "best" practical ellipticity condition to work with.

Our objective in this paper is to propose an ellipticity condition that is natural and sharp – that is, cannot be improved upon – for the convergence of continuous parameter dependent problems and the uniform convergence of associated finite element procedures. We show that this new ellipticity condition is applicable to quite general parameter dependent problems, and specifically those mentioned in the paper, and can be tested for in practice. In the next sections we first consider the generic continuous problem, then the associated discrete problems (typically obtained using finite element methods), and thereafter we briefly comment on the numerical evaluation, in practice, of the developed coercivity condition.

Throughout the paper we use the notation of Refs. [4,5].

## 2. Continuous formulation

In this paper, we are concerned with mixed formulations of the following type.

*Find* $(U^\varepsilon, \Sigma^\varepsilon)$ *in* $\mathscr{V} \times \mathscr{T}^+$ *such that*

$$\begin{cases} A(U^\varepsilon, V) + B(V, \Sigma^\varepsilon) = F(V), & \forall V \in \mathscr{V}, \\ B(U^\varepsilon, \Xi) - \varepsilon^2 D(\Sigma^\varepsilon, \Xi) = 0, & \forall \Xi \in \mathscr{T}^+, \end{cases} \tag{1}$$

where $A$, $B$ and $D$ denote bilinear forms defined on the Sobolev spaces $\mathscr{V}$ and $\mathscr{T}^+$ – with $A$ and $D$ symmetric – $F$ is a linear form defined on $\mathscr{V}$, and $\varepsilon$ represents a small dimensionless model parameter with

$$0 < \varepsilon \leqslant \varepsilon_{\max},$$

and e.g. related to the thickness of the structure in structural mechanics, or to the inverse of the bulk modulus in nearly-incompressible formulations. Note that we use $\varepsilon$ as a superscript in the unknowns of (1) to emphasize that the solution depends on the parameter $\varepsilon$.

Under certain assumptions to be specified, this sequence of solutions parametrized by $\varepsilon$ can be shown to converge – when $\varepsilon$ tends to zero – to the solution of the following limit mixed formulation.

*Find* $(U, \Sigma)$ *in* $\mathscr{V} \times \mathscr{T}$ *such that*

$$\begin{cases} A(U, V) + B(V, \Sigma) = F(V), & \forall V \in \mathscr{V}, \\ B(U, \Xi) = 0, & \forall \Xi \in \mathscr{T}, \end{cases} \tag{2}$$

where $\mathscr{T}$ is another Sobolev space less regular than $\mathscr{T}^+$, namely

$$\mathscr{T}^+ \subset \mathscr{T}, \|\cdot\|_{\mathscr{T}} \leqslant C\|\cdot\|_{\mathscr{T}^+}, \tag{3}$$

and such that $\mathscr{T}^+$ is dense in $\mathscr{T}$, meaning that any element of $\mathscr{T}$ has elements of $\mathscr{T}^+$ arbitrarily close to it, namely,

$$\forall \Xi \in \mathscr{T}, \quad \forall \eta > 0, \ \exists \Xi^\eta \in \mathscr{T}^+ \quad \text{s.t. } \|\Xi - \Xi^\eta\|_{\mathscr{T}} < \eta.$$

In our class of models $\mathscr{T}^+$ will be $L^2$, whereas the larger space $\mathscr{T}$ will be dependent on the specific type of formulation considered. In particular, for plate formulations the space $\mathscr{T}$ can be explicitly characterized, while for shell formulations in general only a definition based on an abstract norm is at hand, see [5]. The complexity of the space $\mathscr{T}$ is seen when considering, specifically, the singularities at corners of skew plates [10] and shells, and boundary layers and internal layers of shells [5,11]. In some other cases such as beam formulations and nearly-incompressible elasticity, the space $\mathscr{T}$ is also (essentially) $L^2$.

Defining the subspace

$$\mathscr{V}_0 = \{V \in \mathscr{V} | B(V, \Xi) = 0, \ \forall \Xi \in \mathscr{T}\}, \tag{4}$$

we can see that the second equation in (2) expresses that the displacement solution $U$ lies in $\mathscr{V}_0$. Hence, the limit mixed formulation considered in (2) corresponds to a *constrained problem*. We then have the following result, as shown in [3,5].

**Proposition 1.** *Assume that*

- *$A$ is symmetric and coercive on $\mathscr{V}_0$, i.e. there exists $\gamma > 0$ such that*

$$A(V, V) \geqslant \gamma \|V\|^2_{\mathscr{V}}, \quad \forall V \in \mathscr{V}_0, \tag{5}$$

- *there exists $\delta > 0$ such that*

$$\inf_{\Xi \in \mathscr{T}, \Xi \neq 0} \sup_{V \in \mathscr{V}, V \neq 0} \frac{B(V, \Xi)}{\|V\|_{\mathscr{V}} \|\Xi\|_{\mathscr{T}}} \geqslant \delta. \tag{6}$$

*Then (2) has a unique solution $(U, \Sigma)$ and this solution satisfies*

$$\|U\|_{\mathscr{V}} + \|\Sigma\|_{\mathscr{T}} \leqslant C\|F\|_{\mathscr{V}'} \tag{7}$$

*for some constant C.*

**Remark 1.** Condition (6) is known as the *continuous inf–sup condition*. This condition and the ellipticity condition (5) appear as *sufficient conditions* for the mixed formulation (2) to be well-posed, but by the Banach theorem it is straightforward to see that they are also *necessary conditions* [12]. Namely, given some bilinear forms $A$ and $B$ with $A$ symmetric positive, in order to have existence and uniqueness of a solution $(U, \Sigma)$ *for any* choice of $F$ in (2), we indeed *must* have the conditions (5) and (6) satisfied. Hence, the existence and uniqueness of a solution $(U, \Sigma)$ is in general only assured *if and only if* the ellipticity and inf–sup conditions are satisfied.

In this paper, our objective will be to obtain sharp conditions to guarantee the convergence of the solutions of the parameter dependent formulation (1), and the uniform convergence of associated discrete solutions. Indeed, it was soon realized in the development of mixed methods that the conditions pertaining to the limit problems – namely, (5) and (6) for the continuous formulation – are not in general sufficient to guarantee the convergence of the parameter dependent problems. Therefore, the classical convergence results have been established under stronger conditions, such as the coercivity of $A$ over the whole space $\mathscr{V}$, or the restrictive assumption $\mathscr{T}^+ = \mathscr{T}$, see [3] and references therein.

However, for some important categories of physical problems such conditions and assumptions do not hold. In particular, for plate and shell formulations $A$ typically represents the bending energy – which does not control the total energy – and we do not have $\mathscr{T}^+ = \mathscr{T}$ as already mentioned. And of course, more generally, obtaining sharp convergence conditions is of fundamental significance, and also of practical interest if these conditions can be tested for specific formulations.

We first define the mapping $\Sigma(\cdot)$ from $\mathscr{V}$ into $\mathscr{T}^+$ such that

$$D(\Sigma(V), \Xi) = B(V, \Xi), \quad \forall \Xi \in \mathscr{T}^+, \tag{8}$$

which is a well-defined continuous linear mapping since $D$ is coercive on $\mathscr{T}^+$. We next define the new bilinear form

$$A_0(U, V) = B(U, \Sigma(V)), \tag{9}$$

which is symmetric since

$$\begin{aligned} A_0(V, U) &= B(V, \Sigma(U)) = D(\Sigma(V), \Sigma(U)) = D(\Sigma(U), \Sigma(V)) \\ &= B(U, \Sigma(V)) = A_0(U, V). \end{aligned}$$

It is also positive since

$$A_0(U, U) = D(\Sigma(U), \Sigma(U)).$$

Now, assuming that (1) has a solution, the second equation can be rewritten as

$$\Sigma^\varepsilon = \varepsilon^{-2} \Sigma(U^\varepsilon). \tag{10}$$

Hence, substituting in the first equation of (1) we obtain

$$A(U^\varepsilon, V) + \varepsilon^{-2} A_0(U^\varepsilon, V) = F(V), \quad \forall V \in \mathscr{V}, \tag{11}$$

namely, we have eliminated $\Sigma^\varepsilon$ from the variational formulation using in essence "static condensation" – see e.g. [2] – but here in the continuous problem. Therefore, the mixed formulation (1) is strictly equivalent to (11) together with the relation (10). Since (11) involves in solid mechanics only displacements (and in fluid mechanics only velocities), the formulation looks like the displacement formulation of solids (velocity formulation of fluids). Clearly, for this problem to be well-posed we need to require the coercivity of the bilinear form " $A + \varepsilon^{-2} A_0$", which holds for all values of $0 < \varepsilon \leqslant \varepsilon_{\max}$ if and only if it holds in particular for $\varepsilon = 1$. In general, this coercivity property is a very natural feature of the displacement-based formulation. We then have the following result.

**Proposition 2.** *Assuming that $A + A_0$ is coercive on $\mathscr{V}$ and that (6) holds, recalling also that $D$ is a symmetric bilinear form bounded and coercive on $\mathscr{T}^+$, the problem (1) has a unique solution $(U^\varepsilon, \Sigma^\varepsilon)$ in $\mathscr{V} \times \mathscr{T}^+$, and this solution satisfies*

$$\|U^\varepsilon\|_{\mathscr{V}} + \|\Sigma^\varepsilon\|_{\mathscr{T}} + \varepsilon \|\Sigma^\varepsilon\|_{\mathscr{T}^+} \leqslant C \|F\|_{\mathscr{V}'}, \tag{12}$$

*for some constant $C$ independent of $\varepsilon$. In addition, $(U^\varepsilon, \Sigma^\varepsilon)$ converges to the solution of (2) , namely,*

$$\|U - U^\varepsilon\|_{\mathscr{V}} + \|\Sigma - \Sigma^\varepsilon\|_{\mathscr{T}} \overset{\varepsilon \to 0}{\to} 0, \tag{13}$$

*and*

$$\varepsilon \|\Sigma^\varepsilon\|_{\mathscr{T}^+} \overset{\varepsilon \to 0}{\to} 0. \tag{14}$$

**Proof.** Since (11) is a so-called penalized formulation, we will use classical results pertaining to such formulations, see e.g. [5]. In order to apply these results, let us first note that

$$\begin{aligned} A_0(V, V) = 0 &\iff D(\Sigma(V), \Sigma(V)) = 0 \\ &\iff \Sigma(V) = 0 \\ &\iff B(V, \Xi) = 0, \quad \forall \Xi \in \mathscr{T}^+ \\ &\iff B(V, \Xi) = 0, \quad \forall \Xi \in \mathscr{T} \\ &\iff V \in \mathscr{V}_0, \end{aligned}$$

where we have used the fact that $\mathscr{T}^+$ is dense in $\mathscr{T}$. Then, taking $V \in \mathscr{V}_0$ as a particular choice of test function in (2), we can see that $U$ is the solution of

$$A(U, V) = F(V), \quad \forall V \in \mathscr{V}_0,$$

with $U$ also in $\mathscr{V}_0$ as already noted, hence this is the candidate limit problem for the penalized formulation. And since $A + A_0$ is coercive on $\mathscr{V}$, classical results (see [5] and references therein) entail that Problem (11) has a unique solution (for every value of $\varepsilon$), satisfying

$$\|U^\varepsilon\|_{\mathscr{V}} \leqslant C \|F\|_{\mathscr{V}'}, \quad \|U - U^\varepsilon\|_{\mathscr{V}} \overset{\varepsilon \to 0}{\to} 0, \quad \varepsilon^{-2} A_0(U^\varepsilon, U^\varepsilon) \overset{\varepsilon \to 0}{\to} 0. \tag{15}$$

Then, defining $\Sigma^\varepsilon$ by (10), we have that $(U^\varepsilon, \Sigma^\varepsilon)$ satisfies (1), hence

$$B(V, \Sigma^\varepsilon) = F(V) - A(U^\varepsilon, V), \quad \forall V \in \mathscr{V},$$

and with (6) we obtain

$$\|\Sigma^\varepsilon\|_{\mathscr{T}} \leqslant C \|F\|_{\mathscr{V}'}.$$

In addition, taking $V = U^\varepsilon$ in (11) we obtain

$$\varepsilon^2 D(\Sigma^\varepsilon, \Sigma^\varepsilon) = \varepsilon^{-2} A_0(U^\varepsilon, U^\varepsilon) \leqslant F(U^\varepsilon) \leqslant C \|F\|^2_{\mathscr{V}'},$$

hence, by using the coercivity of $D$,

$$\varepsilon \|\Sigma^\varepsilon\|_{\mathscr{T}^+} \leqslant C \|F\|_{\mathscr{V}'}.$$

To obtain the convergence of $\Sigma^\varepsilon$ we subtract the first equation of (1) from the first equation of (2), which gives

$$B(V, \Sigma - \Sigma^\varepsilon) = A(U - U^\varepsilon, V), \quad \forall V \in \mathscr{V},$$

and we can again use the inf–sup condition (6) to obtain

$$\|\Sigma - \Sigma^\varepsilon\|_{\mathscr{T}} \leqslant C \|U - U^\varepsilon\|_{\mathscr{V}} \overset{\varepsilon \to 0}{\to} 0.$$

Finally, (14) directly follows from $\varepsilon^2 D(\Sigma^\varepsilon, \Sigma^\varepsilon) = \varepsilon^{-2} A_0(U^\varepsilon, U^\varepsilon)$ and (15). □

**Remark 2.** Of course, when $A$ is coercive on the whole space $\mathscr{V}$, then we automatically have the coercivity of $A + A_0$, since $A_0$ is a positive bilinear form.

## 3. Discrete formulation and convergence analysis

We now introduce Galerkin discretizations of the model-parameter dependent problem (1). Therefore, we consider the following discrete problem.

*Find $(U_h^\varepsilon, \Sigma_h^\varepsilon)$ in $\mathscr{V}_h \times \mathscr{T}_h$ such that*

$$\begin{cases} A(U_h^\varepsilon, V) + B(V, \Sigma_h^\varepsilon) = F(V), & \forall V \in \mathscr{V}_h, \\ B(U_h^\varepsilon, \Xi) - \varepsilon^2 D(\Sigma_h^\varepsilon, \Xi) = 0, & \forall \Xi \in \mathscr{T}_h, \end{cases} \tag{16}$$

where $\mathscr{V}_h$ and $\mathscr{T}_h$ denote finite-dimensional – typically finite element based – subspaces of $\mathscr{V}$ and $\mathscr{T}^+$, respectively. Here the second finite element space $\mathscr{T}_h$ must be a discretization space for $\mathscr{T}^+$ due to the presence of $D$ in the formulation, and then $\mathscr{T}_h$ also gives an admissible discrete space for $\mathscr{T}$, since we have the dense inclusion $\mathscr{T}^+ \subset \mathscr{T}$ (namely, $\mathscr{T}^+$ is more regular).

In this discrete system we can eliminate the unknown $\Sigma_h^\varepsilon$ from the variational formulation – similarly to what we did in the continuous setting – by defining the mapping $\Sigma_h(\cdot)$ from $\mathscr{V}$ into $\mathscr{T}_h$ such that

$$D(\Sigma_h(V), \Xi) = B(V, \Xi), \quad \forall \Xi \in \mathscr{T}_h. \tag{17}$$

This clearly defines a continuous linear mapping, and in fact we have

$$\Sigma_h(V) = \Pi_D^h \Sigma(V), \tag{18}$$

where $\Pi_D^h$ denotes the projection operator onto $\mathscr{T}_h$ for the scalar product associated with the bilinear form $D$. We can also define the bilinear form

$$A_0^h(U, V) = B(U, \Sigma_h(V)), \tag{19}$$

which is again symmetric and positive, and we infer that (16) is equivalent to

$$A(U_h^\varepsilon, V) + \varepsilon^{-2} A_0^h(U_h^\varepsilon, V) = F(V), \quad \forall V \in \mathscr{V}_h, \tag{20}$$

together with

$$\Sigma_h^\varepsilon = \varepsilon^{-2} \Sigma_h(U_h^\varepsilon). \tag{21}$$

We will now be concerned with the convergence of the discrete mixed problem (16) under the assumptions of Proposition 2, namely, assuming $A + A_0$ is coercive, in particular. Considering (20) involving only displacements, we clearly need to have $A + \varepsilon^{-2} A_0^h$ coercive on $\mathscr{V}_h$ for all admissible values of $\varepsilon$ – which is equivalent to $A + A_0^h$ coercive. However, this coercivity property does not follow from the coercivity of $A + A_0$, as it depends on the specific choice of the discrete spaces $\mathscr{V}_h$ and $\mathscr{T}_h$. Hence, this is a condition that we need to require from the discrete spaces, namely,

$$A(V, V) + A_0^h(V, V) \geqslant \gamma \|V\|^2_{\mathscr{V}}, \quad \forall V \in \mathscr{V}_h, \tag{22}$$

for some $\gamma > 0$ *independent of h*. Of course, we will also require the classical discrete inf–sup condition

$$\inf_{\Xi \in \mathscr{T}_h, \Xi \neq 0} \sup_{V \in \mathscr{V}_h, V \neq 0} \frac{B(V, \Xi)}{\|V\|_{\mathscr{V}} \|\Xi\|_{\mathscr{T}}} \geqslant \delta. \tag{23}$$

In order to analyze the convergence under the discrete coercivity condition (22) we will use the auxiliary discrete problem:

*Find* $(\widetilde{U}_h^\varepsilon, \tilde{\Sigma}_h^\varepsilon)$ *in* $\mathscr{V}_h \times \mathscr{T}_h$ *such that*

$$\widetilde{M}_\varepsilon(\widetilde{U}_h^\varepsilon, \tilde{\Sigma}_h^\varepsilon; V, \Xi) = F(V), \quad \forall (V, \Xi) \in \mathscr{V}_h \times \mathscr{T}_h, \tag{24}$$

with

$$\widetilde{M}_\varepsilon(V, \Xi; W, \Gamma) = M_\varepsilon(V, \Xi; W, \Gamma) + \alpha D\big(\Sigma(V) - \varepsilon^2 \Xi, \Sigma(W) - \varepsilon^2 \Gamma\big), \tag{25}$$

for some positive constant $\alpha$, where $M_\varepsilon$ denotes the mixed bilinear form naturally associated with the system (1), namely,

$$M_\varepsilon(V, \Xi; W, \Gamma) = A(V, W) + B(V, \Gamma) + B(W, \Xi) - \varepsilon^2 D(\Xi, \Gamma).$$

We first note that $(U^\varepsilon, \Sigma^\varepsilon)$ satisfies

$$\widetilde{M}_\varepsilon(U^\varepsilon, \Sigma^\varepsilon; V, \Xi) = F(V), \quad \forall (V, \Xi) \in \mathscr{V} \times \mathscr{T}, \tag{26}$$

since we have by construction

$$M_\varepsilon(U^\varepsilon, \Sigma^\varepsilon; V, \Xi) = F(V), \quad \forall (V, \Xi) \in \mathscr{V} \times \mathscr{T},$$

and the additional term in $\widetilde{M}_\varepsilon(U^\varepsilon, \Sigma^\varepsilon; V, \Xi)$ vanishes due to $\Sigma(U^\varepsilon) = \varepsilon^2 \Sigma^\varepsilon$. Therefore, (24) represents a consistent discretization of the continuous mixed formulation. We will now establish the convergence of this discretization procedure.

**Proposition 3.** *Under the assumptions of Proposition 2 and supposing that (23) also holds, then for any given* $\alpha \in ]0, 1/(\varepsilon_{\max})^2[$ *used in the definition of* $\widetilde{M}_\varepsilon$*, for* $0 < \varepsilon \leqslant \varepsilon_{\max}$ *the problem (24) has a unique solution* $(\widetilde{U}_h^\varepsilon, \tilde{\Sigma}_h^\varepsilon)$ *and this solution satisfies*

$$\|U^\varepsilon - \widetilde{U}_h^\varepsilon\|_{\mathscr{V}} + \|\Sigma^\varepsilon - \tilde{\Sigma}_h^\varepsilon\|_{\mathscr{T}} + \varepsilon \|\Sigma^\varepsilon - \tilde{\Sigma}_h^\varepsilon\|_{\mathscr{T}^+} \leqslant C \inf_{V \in \mathscr{V}_h, \Xi \in \mathscr{T}_h} \left\{ \|U^\varepsilon - V\|_{\mathscr{V}} + \|\Sigma^\varepsilon - \Xi\|_{\mathscr{T}} + \varepsilon \|\Sigma^\varepsilon - \Xi\|_{\mathscr{T}^+} \right\}, \tag{27}$$

*for some constant C independent of* $\varepsilon$.

**Proof.** Since consistency is ensured – recall (26) – we can focus on establishing the stability of $\widetilde{M}_\varepsilon$ for the norm

$$\|V, \Xi\|_\varepsilon = \left( \|V\|_{\mathscr{V}}^2 + \|\Xi\|_{\mathscr{T}}^2 + \varepsilon^2 \|\Xi\|_{\mathscr{T}^+}^2 \right)^{\frac{1}{2}}.$$

We decompose the proof into two steps.

(i) *Stability in V and* $\varepsilon\|\Xi\|_{\mathscr{T}^+}$. *Taking* $(W_1, \Gamma_1) = (V, -\Xi)$, *we have*

$$\|W_1, \Gamma_1\|_\varepsilon = \|V, \Xi\|_\varepsilon, \tag{28}$$

and

$$\begin{aligned} \widetilde{M}_\varepsilon(V, \Xi; W_1, \Gamma_1) &= A(V, V) + \alpha D(\Sigma(V), \Sigma(V)) + \varepsilon^2(1 - \alpha\varepsilon^2) D(\Xi, \Xi) \\ &= A(V, V) + \alpha A_0(V, V) + \varepsilon^2(1 - \alpha\varepsilon^2) D(\Xi, \Xi) \\ &\geqslant \gamma_1(\|V\|_{\mathscr{V}}^2 + \varepsilon^2 \|\Xi\|_{\mathscr{T}^+}^2), \end{aligned} \tag{29}$$

using the coercivities of $A + A_0$ and $D$, and the fact that

$$1 - \alpha\varepsilon^2 \geqslant 1 - \alpha(\varepsilon_{\max})^2 > 0.$$

(ii) *Stability in* $\|\Xi\|_{\mathscr{T}}$. We use the discrete inf–sup condition (23) to find $W_2$ in $V_h$ such that

$$\|W_2\|_{\mathscr{V}} = \|\Xi\|_{\mathscr{T}}, \quad B(W_2, \Xi) \geqslant \frac{\delta}{2} \|\Xi\|_{\mathscr{T}}^2,$$

and with $\Gamma_2 = 0$ we have

$$\|W_2, \Gamma_2\|_\varepsilon = \|\Xi\|_{\mathscr{T}} \leqslant \|V, \Xi\|_\varepsilon.$$

This choice gives

$$\widetilde{M}_\varepsilon(V, \Xi; W_2, \Gamma_2) \geqslant \gamma_2 \|\Xi\|_{\mathscr{T}}^2 - C_2(\|V\|_{\mathscr{V}}^2 + \varepsilon^2 \|\Xi\|_{\mathscr{T}^+}^2), \tag{30}$$

and we conclude the stability proof by using convex combinations of the test functions $(W_1, \Gamma_1)$ and $(W_2, \Gamma_2)$. □

Note that, if we eliminate $\tilde{\Sigma}_h^\varepsilon$ from the modified problem (24) we obtain

$$A(\widetilde{U}_h^\varepsilon, V) + \alpha A_0(\widetilde{U}_h^\varepsilon, V) + (\varepsilon^{-2} - \alpha) A_0^h(\widetilde{U}_h^\varepsilon, V) = F(V), \quad \forall V \in \mathscr{V}_h, \tag{31}$$

with

$$\tilde{\Sigma}_h^\varepsilon = \varepsilon^{-2} \Sigma_h(\widetilde{U}_h^\varepsilon). \tag{32}$$

**Remark 3.** With the formulation (31), it is quite straightforward to interpret why stability is more easily obtained in the augmented mixed formulation (24) than in the original mixed formulation associated with (20). Namely, in (31) we retain the coercivity contribution provided by the unperturbed form $A_0$ in the term $A + \alpha A_0$. This strategy is quite natural and has been experimented with, see [13–16,2] and the references therein. We can also say that the additional term introduced in $\widetilde{M}_\varepsilon$ is a "stabilization term". In fact, the modified formulation (24) is an "augmented Lagrangian" formulation as defined and discussed in [17]. However, this construction involves the rather arbitrary numerical factor $\alpha$, and formulations not using such factors are clearly preferable in practice as we mentioned already above.

Using the auxiliary problem (24), we can now prove the convergence of the original discrete mixed problem (16).

**Proposition 4.** *Under the assumptions of Proposition 2 and supposing that (22) and (23) also hold, then (16) has a unique solution* $(U_h^\varepsilon, \Sigma_h^\varepsilon)$ *and this solution satisfies*

$$\|U^\varepsilon - U_h^\varepsilon\|_{\mathscr{V}} + \|\Sigma^\varepsilon - \Sigma_h^\varepsilon\|_{\mathscr{T}} + \varepsilon \|\Sigma^\varepsilon - \Sigma_h^\varepsilon\|_{\mathscr{T}^+} \leqslant C \inf_{V \in \mathscr{V}_h, \Xi \in \mathscr{T}_h} \left\{ \|U^\varepsilon - V\|_{\mathscr{V}} + \|\Sigma^\varepsilon - \Xi\|_{\mathscr{T}} + \varepsilon \|\Sigma^\varepsilon - \Xi\|_{\mathscr{T}^+} \right\}, \tag{33}$$

*for some constant C independent of* $\varepsilon$.

**Proof.** The existence and uniqueness directly follow from the equivalence of (16) with (20) and (21), and from the coercivity of $A + \varepsilon^{-2} A_0^h$ on $\mathscr{V}_h$. As regards convergence, we will take advantage of the convergence result established in Proposition 3 for the particular choice

$$\alpha = (2\varepsilon_{\max})^{-2}.$$

Then (27) holds, and using this auxiliary solution, we will establish the convergence in three steps.

(i) *Convergence in* $U_h^\varepsilon$.
The discrete solution $U_h^\varepsilon$ satisfies (20), while $\widetilde{U}_h^\varepsilon$ instead satisfies

$$\left[A + \alpha A_0 + (\varepsilon^{-2} - \alpha) A_0^h\right](\widetilde{U}_h^\varepsilon, V) = F(V), \quad \forall V \in \mathscr{V}_h.$$

Subtracting the two variational equations yields, for any $V \in \mathscr{V}_h$,

$$\begin{aligned} (A + \varepsilon^{-2} A_0^h)(U_h^\varepsilon - \widetilde{U}_h^\varepsilon, V) &= \alpha (A_0 - A_0^h)(\widetilde{U}_h^\varepsilon, V) \\ &= \alpha \Big[ (A_0 - A_0^h)(U^\varepsilon, V) \\ &\quad + (A_0 - A_0^h)(\widetilde{U}_h^\varepsilon - U^\varepsilon, V) \Big]. \end{aligned}$$

Choosing as a particular test function $V = U_h^\varepsilon - \widetilde{U}_h^\varepsilon$ and using the coercivity (22) with the continuities of $A_0$ and $A_0^h$, we obtain

$$\|U_h^\varepsilon - \widetilde{U}_h^\varepsilon\|_{\mathscr{V}} \leqslant C \left[ \|U^\varepsilon - \widetilde{U}_h^\varepsilon\|_{\mathscr{V}} + \sup_{V \in \mathscr{V}_h} \frac{(A_0 - A_0^h)(U^\varepsilon, V)}{\|V\|_{\mathscr{V}}} \right].$$

Then, a simple triangle inequality gives

$$\|U^\varepsilon - U_h^\varepsilon\|_{\mathscr{V}} \leqslant C\left[\|U^\varepsilon - \widetilde{U}_h^\varepsilon\|_{\mathscr{V}} + \sup_{V\in\mathscr{V}_h} \frac{(A_0 - A_0^h)(U^\varepsilon, V)}{\|V\|_{\mathscr{V}}}\right]. \quad (34)$$

In order to bound the consistency error term in the right-hand side, we use (9) and (19), then (10) and (18) to write

$$(A_0 - A_0^h)(U^\varepsilon, V) = B(V, \Sigma(U^\varepsilon) - \Sigma_h(U^\varepsilon)) = \varepsilon^2 B(V, (I - \Pi_D^h)\Sigma^\varepsilon).$$

Therefore,

$$\frac{|(A_0 - A_0^h)(U^\varepsilon, V)|}{\|V\|_{\mathscr{V}}} \leqslant C\varepsilon^2 \|(I - \Pi_D^h)\Sigma^\varepsilon\|_{\mathscr{T}^+} \leqslant C\varepsilon^2 \inf_{\Xi\in\mathscr{T}_h} \|\Sigma^\varepsilon - \Xi\|_{\mathscr{T}^+},$$

since $D$ provides a norm that is equivalent to $\|\cdot\|_{\mathscr{T}^+}$, and then (34) yields

$$\|U^\varepsilon - U_h^\varepsilon\|_{\mathscr{V}} \leqslant C\left[\|U^\varepsilon - \widetilde{U}_h^\varepsilon\|_{\mathscr{V}} + \varepsilon^2 \inf_{\Xi\in\mathscr{T}_h} \|\Sigma^\varepsilon - \Xi\|_{\mathscr{T}^+}\right]. \quad (35)$$

(ii) *Convergence in* $\|\Sigma_h^\varepsilon\|_{\mathscr{T}}$.
Recalling that $\Sigma^\varepsilon$ satisfies

$$A(U^\varepsilon, V) + B(V, \Sigma^\varepsilon) = F(V), \quad \forall V \in \mathscr{V},$$

we have, for any given $\Gamma \in \mathscr{T}_h$,

$$B(V, \Gamma) = F(V) - A(U^\varepsilon, V) + B(V, \Gamma - \Sigma^\varepsilon), \quad \forall V \in \mathscr{V},$$

while $\Sigma_h^\varepsilon$ satisfies

$$B(V, \Sigma_h^\varepsilon) = F(V) - A(U_h^\varepsilon, V), \quad \forall V \in \mathscr{V}_h.$$

Subtracting the two equations, we obtain

$$B(V, \Sigma_h^\varepsilon - \Gamma) = A(U^\varepsilon - U_h^\varepsilon, V) + B(V, \Sigma^\varepsilon - \Gamma), \quad \forall V \in \mathscr{V}_h.$$

Using the discrete inf–sup condition (23) and the continuities of $A$ and $B$, we infer

$$\|\Sigma_h^\varepsilon - \Gamma\|_{\mathscr{T}} \leqslant C(\|U^\varepsilon - U_h^\varepsilon\|_{\mathscr{V}} + \|\Sigma^\varepsilon - \Gamma\|_{\mathscr{T}}),$$

which yields, combined with a triangle inequality,

$$\|\Sigma^\varepsilon - \Sigma_h^\varepsilon\|_{\mathscr{T}} \leqslant C\left[\|U^\varepsilon - U_h^\varepsilon\|_{\mathscr{V}} + \inf_{\Xi\in\mathscr{T}_h} \|\Sigma^\varepsilon - \Xi\|_{\mathscr{T}}\right]. \quad (36)$$

(iii) *Convergence in* $\varepsilon\|\Sigma_h^\varepsilon\|_{\mathscr{T}^+}$.
We have by construction

$$\varepsilon^2 D(\Sigma^\varepsilon, \Xi) = B(U^\varepsilon, \Xi), \quad \forall \Xi \in \mathscr{T}^+,$$

and

$$\varepsilon^2 D(\Sigma_h^\varepsilon, \Xi) = B(U_h^\varepsilon, \Xi), \quad \forall \Xi \in \mathscr{T}_h.$$

From this we infer, for any given $\Gamma \in \mathscr{T}_h$,

$$\varepsilon^2 D(\Sigma_h^\varepsilon - \Gamma, \Sigma_h^\varepsilon - \Gamma) = B(U_h^\varepsilon - U^\varepsilon, \Sigma_h^\varepsilon - \Gamma) + \varepsilon^2 D(\Sigma^\varepsilon - \Gamma, \Sigma_h^\varepsilon - \Gamma),$$

and – recalling the coercivity of $D$ on $\mathscr{T}^+$ – standard manipulations then give

$$\varepsilon\|\Sigma_h^\varepsilon - \Gamma\|_{\mathscr{T}^+} \leqslant C(\|U^\varepsilon - U_h^\varepsilon\|_{\mathscr{V}} + \|\Sigma_h^\varepsilon - \Gamma\|_{\mathscr{T}} + \varepsilon\|\Sigma^\varepsilon - \Gamma\|_{\mathscr{T}^+}),$$

hence, with the help of triangle inequalities,

$$\varepsilon\|\Sigma^\varepsilon - \Sigma_h^\varepsilon\|_{\mathscr{T}^+} \leqslant C\Big(\|U^\varepsilon - U_h^\varepsilon\|_{\mathscr{V}} + \|\Sigma^\varepsilon - \Sigma_h^\varepsilon\|_{\mathscr{T}} + \inf_{\Xi\in\mathscr{T}_h} \{\|\Sigma^\varepsilon - \Xi\|_{\mathscr{T}} + \varepsilon\|\Sigma^\varepsilon - \Xi\|_{\mathscr{T}^+}\}\Big). \quad (37)$$

In order to obtain the final error estimate (33), it now just remains to gather the intermediate bounds (27), (35), (36) and (37). □

**Remark 4.** Note that we have used the solution of the discrete stabilized formulation (24) only as an *intermediary* in the convergence proof, and that no such stabilization is actually included in the mixed formulation (16).

**Remark 5.** Regarding the coercivity assumption (22), we note that

$$A_0^h(V, V) = D(\Sigma_h(V), \Sigma_h(V)) = D(\Pi_D^h\Sigma(V), \Pi_D^h\Sigma(V)),$$

while $A_0$, which provides the desired coercivity in $A + A_0$, satisfies instead

$$A_0(V, V) = D(\Sigma(V), \Sigma(V)).$$

Hence, in order for $A + A_0^h$ to remain coercive, we need to have $\mathscr{T}_h$ "sufficiently large" to avoid losing the coercivity in the projection $\Pi_D^h$. However, this may be difficult to accommodate with the inf–sup condition (23), which is more easily satisfied when $\mathscr{T}_h$ is "sufficiently small". Nevertheless, we point out that the coercivity assumption (22) can be numerically tested by computing the smallest eigenvalue in the eigenproblem

$$A(\Phi, V) + A_0^h(\Phi, V) = \lambda(\Phi, V)_{\mathscr{V}}, \quad \forall V \in \mathscr{V}_h.$$

In the first instance, a single element should not display a zero eigenvalue (spurious mode), but the same should also hold for increasingly refined meshes, like in the numerical inf–sup test presented in [18].

In some cases, we will have that $\mathscr{T}^+ = \mathscr{T}$. This holds, in particular, for nearly-incompressible formulations where $\mathscr{T}$ and $\mathscr{T}^+$ correspond to $L^2$ (with the mean value of the function subtracted in the norm of $\mathscr{T}$ when considering homogeneous boundary conditions all over the boundary for the displacements, since the pressure is then defined up to a constant), see [2,3]. Another such example is the mixed formulation corresponding to the Timoshenko beam model, see [2,5]. In such cases the convergence analysis can be directly performed – see in particular [3,19] – but it is quite illuminating to see that it can also be inferred from our above discussions. Hence, we will show that in this case the partial coercivity (5) implies the coercivity of $A + A_0$, by which we have the following well-posedness and convergence result.

**Proposition 5.** *Consider the case when* $\mathscr{T}^+ = \mathscr{T}$. *Assume that* (5) *and* (6) *both hold, that the bilinear form A is positive, namely,*

$$A(V, V) \geqslant 0, \quad \forall V \in \mathscr{V}, \quad (38)$$

*and that D is a symmetric bilinear form coercive on* $\mathscr{T}$. *Then the problem* (1) *has a unique solution and this solution satisfies*

$$\|U^\varepsilon\|_{\mathscr{V}} + \|\Sigma^\varepsilon\|_{\mathscr{T}} \leqslant C\|F\|_{\mathscr{V}'}, \quad (39)$$

*for some constant C independent of* $\varepsilon$. *In addition,* $(U^\varepsilon, \Sigma^\varepsilon)$ *converges to the solution of* (2) *, namely,*

$$\|U - U^\varepsilon\|_{\mathscr{V}} + \|\Sigma - \Sigma^\varepsilon\|_{\mathscr{T}} \overset{\varepsilon\to 0}{\to} 0. \quad (40)$$

**Proof.** It suffices to show that $A + A_0$ is coercive in this case, in order to apply Proposition 2. For any $V \in \mathscr{V}$ we have

$$A_0(V, V)^{\frac{1}{2}} = D(\Sigma(V), \Sigma(V))^{\frac{1}{2}} = \sup_{\Xi\in\mathscr{T}} \frac{D(\Sigma(V), \Xi)}{D(\Xi, \Xi)^{\frac{1}{2}}} = \sup_{\Xi\in\mathscr{T}} \frac{B(V, \Xi)}{D(\Xi, \Xi)^{\frac{1}{2}}},$$

by the definition of $\Sigma(V)$, recall (8). Of course, we have $D(\Xi, \Xi)^{\frac{1}{2}} \leqslant C\|\Xi\|_{\mathscr{T}^+}$, and we now use the fact that $\mathscr{T}^+ = \mathscr{T}$ to infer

$$A_0(V, V)^{\frac{1}{2}} \geqslant C \sup_{\Xi\in\mathscr{T}} \frac{B(V, \Xi)}{\|\Xi\|_{\mathscr{T}}}. \quad (41)$$

On the other hand, we have

$$\mathscr{V} = \mathscr{V}_0 \oplus \mathscr{V}_0^\perp,$$

and for any $V \in \mathscr{V}$ we can write the decomposition

$$V = V_0 + V_1, \quad V_0 \in \mathscr{V}_0, \quad V_1 \in \mathscr{V}_0^{\perp}.$$

Then

$$(A + A_0)(V,V) = A(V_0,V_0) + A(V_1,V_1) + 2A(V_0,V_1) + A_0(V_1,V_1),$$

and we can combine (41) with the classical alternative inf–sup bound (see [3,5])

$$\inf_{V \in \mathscr{V}_0^{\perp}, V \neq 0} \sup_{\Xi \in \mathscr{T}, \Xi \neq 0} \frac{B(V,\Xi)}{\|V\|_{\mathscr{V}} \|\Xi\|_{\mathscr{T}}} \geqslant \delta, \tag{42}$$

to obtain

$$A_0(V_1,V_1) \geqslant \gamma \|V_1\|^2_{\mathscr{V}}.$$

Furthermore, since $A$ is positive we can apply the Cauchy–Schwarz inequality and infer

$$(A + A_0)(V,V) \geqslant (1-\eta)A(V_0,V_0) + (1 - 1/\eta)A(V_1,V_1) + \gamma \|V_1\|^2_{\mathscr{V}},$$

for any $0 < \eta < 1$. Then we can choose $\eta$ so that

$$(1 - 1/\eta)A(V_1,V_1) + \gamma\|V_1\|^2 \geqslant \frac{\gamma}{2}\|V_1\|^2_{\mathscr{V}},$$

and since $A$ is coercive on $\mathscr{V}_0$ we obtain the desired coercivity of $A + A_0$. $\square$

**Remark 6.** The assumption $\mathscr{T}^+ = \mathscr{T}$ was crucially used in this proof to obtain (41). In the general case, we only have

$$\|\Xi\|_{\mathscr{T}} \leqslant C\|\Xi\|_{\mathscr{T}^+}, \quad \forall \Xi \in \mathscr{T},$$

and (41) cannot be inferred from the previous equation.

We can then show the following result for corresponding finite element discretizations.

**Proposition 6.** *Under the assumptions of Proposition 5 and supposing that (23) also holds and that $A$ is coercive on*

$$\mathscr{V}_{h0} = \{V \in \mathscr{V}_h | B(V,\Xi) = 0, \quad \forall \Xi \in \mathscr{T}_h\},$$

*then Problem (16) has a unique solution $(U_h^{\varepsilon}, \Sigma_h^{\varepsilon})$. Moreover this solution satisfies, for $0 < \varepsilon \leqslant \varepsilon_{\max}$,*

$$\|U^{\varepsilon} - U_h^{\varepsilon}\|_{\mathscr{V}} + \|\Sigma^{\varepsilon} - \Sigma_h^{\varepsilon}\|_{\mathscr{T}} \leqslant C \inf_{V \in \mathscr{V}_h, \Xi \in \mathscr{T}_h} \{\|U^{\varepsilon} - V\|_{\mathscr{V}} + \|\Sigma^{\varepsilon} - \Xi\|_{\mathscr{T}}\}, \tag{43}$$

for some constant $C$ independent of $\varepsilon$.

**Proof.** Similarly to the proof of Proposition 5, we will simply show that $A + A_0^h$ is coercive on $\mathscr{V}_h$ in order to apply Proposition 4, and we just sketch this proof. We first directly obtain the discrete counterpart of (41), namely,

$$A_0^h(V,V)^{\frac{1}{2}} \geqslant C \sup_{\Xi \in \mathscr{T}_h} \frac{B(V,\Xi)}{\|\Xi\|_{\mathscr{T}}}, \quad \forall V \in \mathscr{V}_h, \tag{44}$$

using similar arguments. Then, the coercivity proof can be concluded by employing the decomposition

$$\mathscr{V}_h = \mathscr{V}_{h0} \oplus \mathscr{V}_{h0}^{\perp} = \mathscr{V}_{h0} \oplus \mathscr{V}_{h1},$$

and the alternate discrete inf–sup condition (see [3,5])

$$\inf_{V \in \mathscr{V}_{h0}^{\perp}, V \neq 0} \sup_{\Xi \in \mathscr{T}_h, \Xi \neq 0} \frac{B(V,\Xi)}{\|V\|_{\mathscr{V}} \|\Xi\|_{\mathscr{T}}} \geqslant \delta, \tag{45}$$

combined with (44) to show that

$$A_0^h(V_1,V_1) \geqslant \gamma\|V_1\|^2_{\mathscr{V}}, \quad \forall V_1 \in \mathscr{V}_{h1}. \qquad \square$$

## 4. Remarks on numerical assessment

In practice, when a formulation (1) is solved with a discretization scheme using (16), it is expedient to employ static condensation on the variables $\Sigma_h^{\varepsilon}$. This results into the governing equation (20). The condition is then that the smallest eigenvalue of this problem (for any $h$) be bounded from below.

We note that this coercivity condition is required for all problems governed by formulation (1), that is, irrespective of whether the space $\mathscr{T}$ or $\mathscr{T}^+$ is applicable, and in the development of new finite element discretization schemes, this condition can easily be tested for.

As examples, considering the displacement–pressure formulation of almost incompressible 2D analysis of solids, the discretization based on elements with nine nodes for the displacements and a linear pressure assumption (the 9/3 element or Q2/P1 element [2,3]) of course satisfies the condition.

However, using an axisymmetric element with the usual eight nodes for the displacement interpolation and a constant pressure assumption (the 8/1 element or Q2'/P0 element) this ellipticity condition is violated. For example, a single element test shows that we obtain a zero eigenvalue by considering the displacements

$$v = crs + 3cs, \quad w = -3cr - \frac{c}{2}(r^2 - s^2 + 1),$$

where $(r,s)$ denote the local coordinate system of the $2 \times 2$ element with the element center located at radius $R = 3$, $(v,w)$ are the corresponding displacement components, and $c$ is a constant. Note that, regarding Proposition 6, in this case the inf–sup condition (23) is passed but $A$ is not coercive on $\mathscr{V}_{h0}$.

Considering the analysis of plates and shells, the formulations of the various MITC elements that have been proposed have always been tested for this coercivity condition, see e.g. [20–22], and a general single element test has so far been found sufficient.

## 5. Conclusions

We proposed in this paper – for continuous model-parameter dependent problems – a sharp ellipticity condition which when satisfied together with the inf–sup condition guarantees the convergence of the formulation for the parameter approaching zero. The ellipticity condition is quite natural, and for associated discrete formulations, if the corresponding ellipticity and inf–sup conditions hold, uniform convergence of the discretization procedure with best approximation properties is obtained.

An important point is that this approach of analyzing the quality of discretization schemes is quite general in that it does not depend on the space in which the constraint variables are measured ($\mathscr{T}$ or $\mathscr{T}^+$). For optimal approximation quality, the inf–sup condition and given ellipticity condition need be satisfied.

The proposed condition can also be tested for easily in practice – and should be tested for – when developing new discretization schemes.

Finally, it is rather natural to conjecture that this ellipticity condition is also applicable in dynamic analysis in order to have – together with the inf–sup condition – sufficient conditions to ensure the uniform and optimal convergence of the eigenvalues and eigenspaces of the associated discretized spectral problem, thereby extending the results of [23].

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