On the flow-condition-based interpolation approach for incompressible fluids

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Abstract

We briefly review the use of the flow-condition-based interpolation (FCBI) approach for the solution of high Péclet and high Reynolds number flow problems. The basic idea is presented, formulations for quadrilateral and triangular elements are briefly described, and the results of some numerical solutions are given.

Keywords: Incompressible flows; Stabilization; Flow-condition-based interpolation; FCBI approach

1. Introduction

While much research has been expended on the development of numerical schemes for the solution of high Péclet (Pe) and high Reynolds (Re) number flow problems, there is still need to reach more stable and more accurate procedures. Our approach for the development of solution schemes for high Pe and Re flow problems has been presented in [1,2], and consists of seeking procedures that, firstly, are stable and give reasonable solutions in laminar flows even when using rather coarse meshes and, secondly, give optimal accuracy in laminar and turbulent flow assumptions. In practice, it can be effective that laminar flow solutions are sought first, in which stability and optimal accuracy are required with rather coarse meshes, after which finer meshes and appropriate turbulence models are selected. This approach will be particularly valuable when used together with goal-oriented error measures, and in the aim to employ rather coarse meshes in the solution of fluid–structure interaction problems [3]. Stability and accuracy are obtained by using a flow-condition-based interpolation and control volumes in the finite element solution. Hence, flow-condition-based interpolation (FCBI) discretization procedures represent hybrid methods of classical finite element and finite volume techniques. While FCBI methods are already used in engineering practice, we aim to improve the schemes.

The objective of this paper is to briefly summarize some recent experiences with the flow-condition-based interpolation approach for the solution of high Pe and high Re number flows. We consider two-dimensional flow conditions and first present a technique for the advection–diffusion problem, using quadrilateral elements, and then a technique for the incompressible Navier–Stokes equations, using triangular elements. The use of triangular elements in unstructured meshes is of much interest but, in particular, still needs to be further developed, and the FCBI approach described here has considerable potential.

2. FCBI methods for the advection–diffusion problem

In this section we consider steady-state incompressible flow advection–diffusion problems to be solved using quadrilateral grids. We assume that the problems are well-posed in the Hilbert space Θ. For the finite element solution, we use a Petrov-Galerkin variational formulation with subspaces Φ_h, Θ_h and W_h of Θ. The non-dimensional finite element formulation is [1,4]:

Find the temperature $\phi \in \Phi_h$, $\theta \in \Theta_h$ such that, for all $w \in W_h$,

$$\int_{\Omega} w\nabla \cdot \left(\mathbf{v}\phi - \frac{1}{Pe}\nabla\theta\right) d\Omega = 0 \qquad (1)$$

where $\mathbf{v}$ is the prescribed velocity, Pe is the Péclet number and $\Omega \in \Re^2$ is a domain with the boundary $S = \bar{S}_\theta$. For the notation used, see [1,4,5].

Figure 1(a) shows a mesh of elements, and an element

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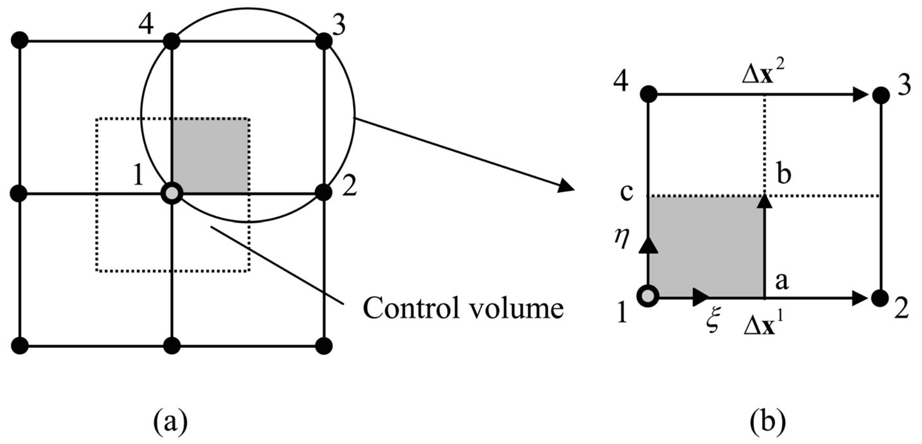

Fig. 1. Control volume on the four-node element: (a) four-node element; (b) section ab.

in its natural coordinate system. Here, both velocity and temperature are defined at each node of the four-node elements. The weight functions in the space W_h are step functions. While the trial functions in Θ_h are the bilinear interpolation functions, the trial functions in Φ_h are, in the original FCBI scheme [1], defined considering the flow conditions along each side of the element in order to reach a stable solution. The functions are for the flux through ab in Fig. 1(b),

$$\begin{bmatrix} h_1^\phi & h_4^\phi \\ h_2^\phi & h_3^\phi \end{bmatrix} = \left[\mathbf{h}\left(x^1\right), \mathbf{h}\left(x^2\right)\right]\mathbf{h}(\eta)\mathbf{h}^{\mathrm{T}}(\eta) \tag{2}$$

with

$$x^k = \frac{e^{q^k\xi} - 1}{e^{q^k} - 1}, \quad q^k = Pe\left(\bar{\mathbf{v}}^k \cdot \Delta\mathbf{x}^k\right) \tag{3}$$

where $\mathbf{h}^{\mathrm{T}}(y) = [1 - y, y]$ ($y = x^1, x^2$ with $0 \le \xi, \eta \le 1$), $\bar{\mathbf{v}}^k$ is the average velocity evaluated at the center of the sides considered ($\xi = 1/2$ and $\eta = 0$, 1 for $k = 1$, 2, respectively) and is calculated using the nodal velocity variables $\mathbf{v}_i$. Similarly, the functions for the flux through bc are obtained by using the flow conditions along the corresponding sides.

For some applications, more effective trial functions in Φ_h can be established by considering directly two-dimensional flow conditions in the interpolations. The proposed functions are written as follows:

$$\begin{bmatrix} h_1^\phi & h_4^\phi \\ h_2^\phi & h_3^\phi \end{bmatrix} = \mathbf{h}(\alpha)\mathbf{h}^{\mathrm{T}}(\beta) \tag{4}$$

Equation (4) corresponds to a two-dimensional general solution in a rectangular element when we choose the one-dimensional exact solution based interpolations in the ξ and η directions for $\mathbf{h}(\alpha)$ and $\mathbf{h}(\beta)$, respectively. As an alternative, link-cutting bubbles [6] can be introduced to obtain $\mathbf{h}(\alpha)$ and $\mathbf{h}(\beta)$ in a similar way [4]. Note that in all above methods, the bilinear interpolation functions are obtained when the element Péclet numbers approach zero.

3. FCBI scheme for triangular grids

In this section we present an FCBI method using triangular grids for the steady-state analysis of incompressible flows. We assume that the problem is well-posed in the Hilbert spaces V and P. As in the advection–diffusion problem, we use a Petrov-Galerkin variational formulation; the subspaces are now U_h, V_h and W_h of V, and P_h and Q_h of P for the finite element solution of the Navier–Stokes equations. The non-dimensional finite element formulation used is:

Find the velocity $\mathbf{u} \in U_h$, $\mathbf{v} \in V_h$ and $p \in P_h$ such that for all $w \in W_h$ and $q \in Q_h$

$$\int_\Omega w\nabla \cdot \left[\mathbf{uv} + p\mathbf{I} - \frac{1}{Re}\left\{\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right\}\right]d\Omega = \mathbf{0} \tag{5}$$

$$\int_\Omega q\nabla \cdot \mathbf{u}d\Omega = 0 \tag{6}$$

where p is the pressure, $\mathbf{I}$ is the identity tensor, Re is the Reynolds number and $\Omega \in \Re^2$ is a domain with the boundary $S = \bar{S}_v \cup S_f\,(S_v \cap S_f = \emptyset)$. The trial functions in U_h and P_h are the usual functions of finite element interpolations for velocity and pressure, respectively. These are selected to satisfy the inf-sup condition of incompressible analysis [7]. The trial functions in V_h are defined using the flow conditions. The weight functions in the spaces W_h and Q_h are step functions, which

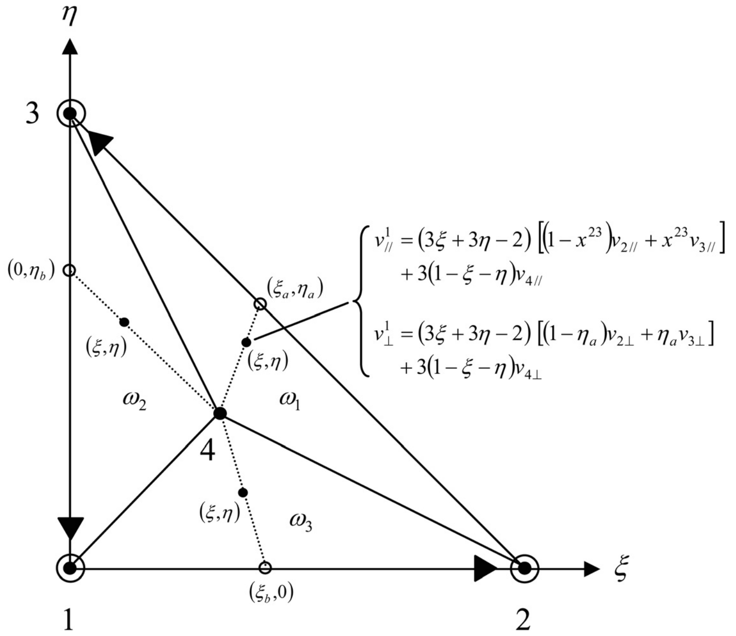

Fig. 2. Illustration of the interpolation of velocity **v** (see [8]).

enforce the local conservation of momentum and mass, respectively.

The proposed procedure employs the MINI element shown in Fig. 2, in which the velocity is interpolated using all four nodes, the local node numbers 1 to 4, while the pressure is calculated by linear interpolation using only the three corner nodes, the local node numbers 1 to 3. The velocity **u** is interpolated with the linear functions plus a bubble function [5], and the velocity **v** in the advection term is interpolated using the analytical solution of the one-dimensional flow equation along the sides of the element. The functions for **v** in the domain ω_1 are indicated in Fig. 2. The important feature of the trial functions is that the flow conditions are used on all three sides of the element. Consequently, an interpolated value at a specific point does not depend on the node numbering. Of course, as the element Reynolds numbers become small, the trial functions in V_h approach the functions in U_h. More details regarding the scheme are given in [8].

4. Numerical examples

4.1. Temperature solution in flow across a square domain

In order to test the proposed FCBI schemes for advection–diffusion problems, we consider the temperature solution in a flow with a skew advective velocity. Figure 3(a) shows the definition of the problem including the Dirichlet boundary conditions. The constant unit advective velocity is prescribed over the complete domain. A rather coarse mesh of 12 × 12 square elements is used for the solution of the temperature when $Pe = 10^6$. Figures 3(b) and 3(c) show the results obtained with the FCBI methods based on a general solution and link-cutting bubbles, respectively. The results indicate the stability and accuracy of the schemes.

4.2. Solution of the driven cavity flow problem

The capability of the FCBI method using triangular grids is illustrated by solving the lid-driven cavity flow problem, in which the no-slip boundary condition is imposed on the left, lower and right boundaries of the square domain, while a unit velocity is prescribed on the upper boundary. Figure 4(a) gives the mesh of 20 × 20 × 2 elements used for the solution of the flow when $Re = 1000$. As shown in Fig. 4, for this rather coarse mesh, a reasonably accurate solution is obtained with the proposed method. In Fig. 4(c) the obtained results are compared with the solution of Ghia et al. [9]. We also note that the velocity profiles become close to those of Ghia et al. when the number of elements per side is doubled, i.e. a mesh of 40 × 40 × 2 elements is used.

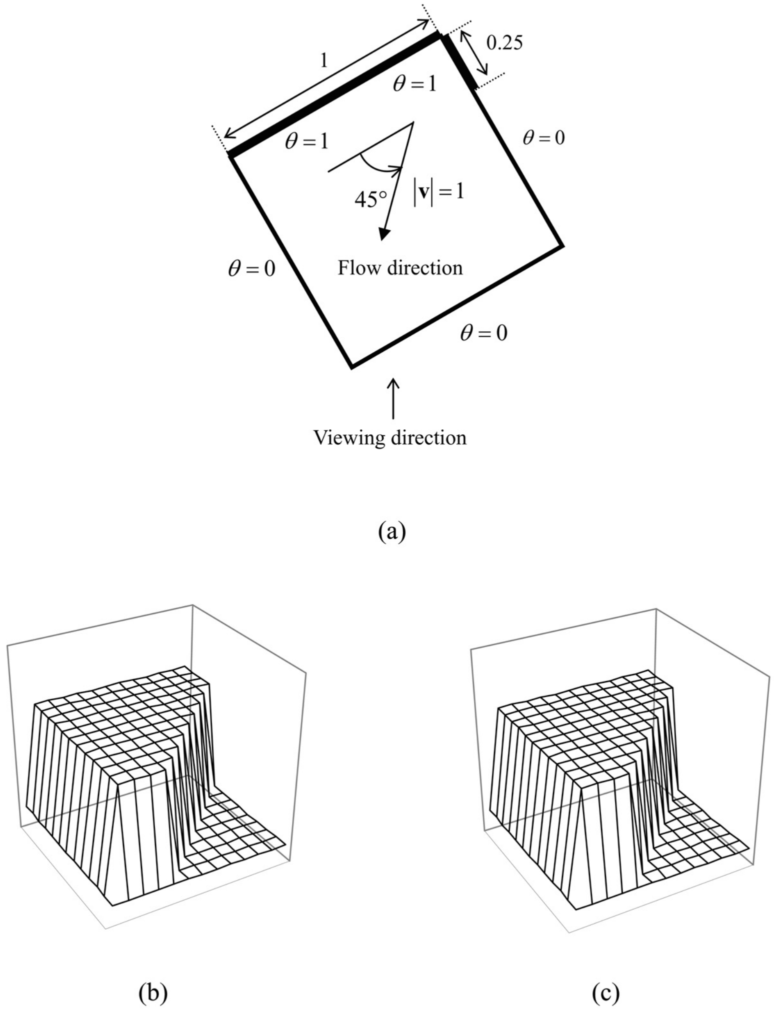

Fig. 3. Temperature solutions for $Pe = 10^6$ in the flow problem with skew advective velocity: (a) problem definition; (b) solution using FCBI scheme based on a general solution; (c) solution using FCBI scheme based on link-cutting bubbles.

5. Conclusions

The objective in the FCBI solution approach is to have solution schemes that are stable at high Pe and Re numbers, and give reasonable solutions, even when using rather coarse meshes. Using such schemes together with error measures, rather coarse meshes can be employed to solve certain fluid flow problems in engineering practice, and notably fluid-flow structural interaction problems.

We summarized in this paper some recent experiences and developments with the FCBI solution approach in two-dimensional analyses; but of course, three-dimensional analyses can and are already widely performed based on this approach [2,10]. While there is such use with first- and second-order FCBI schemes, we still continue to seek more effective FCBI procedures.

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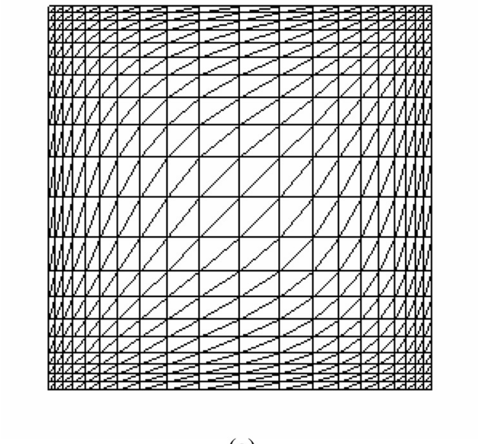
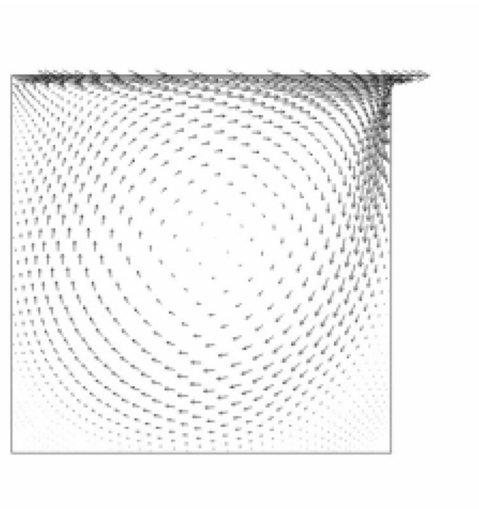


(a)



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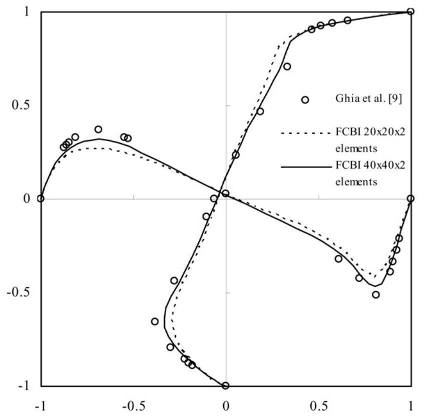

(c)

Fig. 4. Driven cavity flow problem: (a) mesh; (b) velocity distribution; (c) comparison of vertical and horizontal velocity profiles along the centerlines for $Re = 1000$.

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