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The Bathe time integration method revisited for prescribing desired numerical dissipation

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ABSTRACT

In this paper we further consider the Bathe method for the direct time integration in structural dynamics and wave propagations. The method uses two sub-steps per time step and is an unconditionally stable scheme frequently used without adjusting any parameter. In the first sub-step the trapezoidal rule is used and in the second sub-step the 3-point Euler backward method is employed. In this contribution we derive the method using, instead of the Euler scheme, the 3-point trapezoidal rule for the complete step with two Newmark parameters. The parameters can then be used to smoothly prescribe desired numerical dissipation, from zero to very significant dissipation. To highlight the performance of the method, the stability, accuracy and overshooting are studied and some illustrative problems are solved. The results are compared with those of some other methods that also use parameters to introduce numerical dissipation. We conclude that the use of the parameters in the Bathe method can be valuable but probably will require some numerical experimentation.

1. Introduction

A great deal of research has been performed to solve the finite element equations of motion using direct time integration methods, and different explicit and implicit algorithms have been proposed. Some well-known implicit methods are the Newmark [1], Wilson [2], HHT schemes [3] and the three-parameter method also known as the generalized alpha method [4,5], all of which use adjustable parameters to control the numerical dissipation. However, these methods have shortcomings in accuracy and hence more recently, in Refs. [6–8] a time integration method, referred to as the Bathe method, was introduced for linear and nonlinear analyses. The procedure is an implicit time integration method with unconditional stability that uses two sub-steps for each time step. The use of the intermediate time solution can be interpreted as simply "some step internal computations" to reach the final solution [9].

In the Bathe procedure, we use for the first sub-step, the trapezoidal rule and for the second sub-step the 3-point Euler backward method. While, in principle, three parameters can be varied, namely the two parameters (δ, α) in the Newmark method and one parameter, γ, giving the sub-step size, usually the default values are used, $\delta = 1/2, \alpha = 1/4, \gamma = 1/2$. The use of these parameters has been advocated as a strength of the method for practical analyses [7], because in engineering practice adjusting parameters for a solution is hardly desirable. However, in some analyses, an analyst may wish to introduce less or more numerical dissipation and this effect can be achieved to some extent by changing the parameter γ [10,11]. Below we refer to the scheme using $\delta = 1/2, \alpha = 1/4$ with any γ as the "standard" Bathe method. This procedure does not have any overshooting in displacements and velocities, like observed with other techniques [12,13]. In Ref. [14] the authors use the idea of two equal sub-steps, give a scheme developed using a weighted residual formulation and compare it with the standard Bathe method when used with $\gamma = 0.5$.

Our objective in this paper is to look at the Bathe method from a different perspective and thus be able to introduce in a novel manner desired numerical amplitude decay (dissipation) and period elongation (dispersion). We use the trapezoidal rule for the first sub-step and instead of the 3-point Euler scheme for the second sub-step, we use a 3-point trapezoidal rule with the parameters β_1 and β_2. The two parameters are like the δ value in the Newmark method [15] and can be used to smoothly control the desired numerical amplitude decay and period elongation. We refer to the scheme as the β_1/β_2-Bathe method since the parameters can be set to obtain the standard Bathe time integration procedure. In the following we present the governing equations, give the stability and accuracy properties when compared with other schemes and solve some illustrative problems. We conclude that

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the β_1/β_2-Bathe method can be attractive when the analyst desires to control the numerical dissipation to obtain better accuracy in a solution.

2. The β_1/β_2-Bathe time integration scheme

In this section we derive the governing equations of the β_1/β_2-Bathe time integration method and study the stability and accuracy of the scheme. We introduce Newmark-like parameters in the second sub-step formula of the Bathe time integration scheme to be able to prescribe different numerical dissipation.

For the first sub-step of size $\gamma\Delta t$ the trapezoidal rule is used (as in the standard Bathe method)

$$ {}^{t+\gamma\Delta t}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + \left(\frac{\gamma\Delta t}{2}\right)\left({}^{t}\ddot{\mathbf{U}} + {}^{t+\gamma\Delta t}\ddot{\mathbf{U}}\right) \tag{1} $$

$$ {}^{t+\gamma\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + \left(\frac{\gamma\Delta t}{2}\right)\left({}^{t}\dot{\mathbf{U}} + {}^{t+\gamma\Delta t}\dot{\mathbf{U}}\right) \tag{2} $$

which gives

$$ {}^{t+\gamma\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + (\gamma\Delta t)\,{}^{t}\dot{\mathbf{U}} + \left(\frac{\gamma\Delta t}{2}\right)^2\left({}^{t}\ddot{\mathbf{U}} + {}^{t+\gamma\Delta t}\ddot{\mathbf{U}}\right) \tag{3} $$

The equations of motion in linear analysis are at time $t+\gamma\Delta t$

$$ \mathbf{M}\,{}^{t+\gamma\Delta t}\ddot{\mathbf{U}} + \mathbf{C}\,{}^{t+\gamma\Delta t}\dot{\mathbf{U}} + \mathbf{K}\,{}^{t+\gamma\Delta t}\mathbf{U} = {}^{t+\gamma\Delta t}\mathbf{R} \tag{4} $$

where $\mathbf{U}$, $\mathbf{R}$ represent the vector of nodal displacements/rotations and the load vector, respectively, an overdot denotes a time derivative, $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ is the damping matrix and $\mathbf{K}$ is the stiffness matrix.

Using Eqs. (1)–(3) in Eq. (4) we obtain

$$ \widehat{\mathbf{K}}_1\,{}^{t+\gamma\Delta t}\mathbf{U} = {}^{t+\gamma\Delta t}\widehat{\mathbf{R}}_1 \tag{5} $$

where

$$ \widehat{\mathbf{K}}_1 = \frac{4}{(\gamma\Delta t)^2}\mathbf{M} + \frac{2}{\gamma\Delta t}\mathbf{C} + \mathbf{K} \tag{6} $$

$$ {}^{t+\gamma\Delta t}\widehat{\mathbf{R}}_1 = {}^{t+\gamma\Delta t}\mathbf{R} + \mathbf{M}\left(\frac{4}{(\gamma\Delta t)^2}{}^{t}\mathbf{U} + \frac{4}{\gamma\Delta t}{}^{t}\dot{\mathbf{U}} + {}^{t}\ddot{\mathbf{U}}\right) + \mathbf{C}\left(\frac{2}{\gamma\Delta t}{}^{t}\mathbf{U} + {}^{t}\dot{\mathbf{U}}\right) \tag{7} $$

For the second sub-step, instead of using the 3-point Euler backward method, we now use the Newmark scheme for the three time points employed and use β_1 and β_2 as the parameters for the sub-steps [15]

$$ \begin{aligned} {}^{t+\Delta t}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + (\gamma\Delta t)\left((1-\beta_1){}^{t}\ddot{\mathbf{U}} + \beta_1{}^{t+\gamma\Delta t}\ddot{\mathbf{U}}\right) + ((1-\gamma)\Delta t) \\ \times\left((1-\beta_2){}^{t+\gamma\Delta t}\ddot{\mathbf{U}} + \beta_2{}^{t+\Delta t}\ddot{\mathbf{U}}\right) \end{aligned} \tag{8} $$

$$ \begin{aligned} {}^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + (\gamma\Delta t)\left((1-\beta_1){}^{t}\dot{\mathbf{U}} + \beta_1{}^{t+\gamma\Delta t}\dot{\mathbf{U}}\right) + ((1-\gamma)\Delta t) \\ \times\left((1-\beta_2){}^{t+\gamma\Delta t}\dot{\mathbf{U}} + \beta_2{}^{t+\Delta t}\dot{\mathbf{U}}\right) \end{aligned} \tag{9} $$

Of course, using Eqs. (8) and (9) with $\beta_1=\beta_2=1/2$ we have the trapezoidal rule, like in the Newmark method [1,15]. It is also easy to show that with $\beta_1=1/3$, $\beta_2=2/3$ and $\gamma=0.5$ the Bathe method of time integration is obtained when $\gamma=0.5$ [6–8,11,15], see Appendix A. The vectors ${}^{t+\Delta t}\ddot{\mathbf{U}}$ and ${}^{t+\Delta t}\dot{\mathbf{U}}$ are given as

$$ \begin{aligned} {}^{t+\Delta t}\ddot{\mathbf{U}} = \frac{1}{(1-\gamma)(\Delta t)\beta_2}\Big[{}^{t+\Delta t}\dot{\mathbf{U}} - {}^{t}\dot{\mathbf{U}} - ((\gamma\Delta t)(1-\beta_1)){}^{t}\ddot{\mathbf{U}} - ((\gamma\Delta t)\beta_1 \\ +(1-\gamma)(\Delta t)(1-\beta_2)){}^{t+\gamma\Delta t}\ddot{\mathbf{U}}\Big] \end{aligned} \tag{10} $$

$$ \begin{aligned} {}^{t+\Delta t}\dot{\mathbf{U}} = \frac{1}{(1-\gamma)(\Delta t)\beta_2}\Big[{}^{t+\Delta t}\mathbf{U} - {}^{t}\mathbf{U} - ((\gamma\Delta t)(1-\beta_1)){}^{t}\dot{\mathbf{U}} - ((\gamma\Delta t)\beta_1 \\ +(1-\gamma)(\Delta t)(1-\beta_2)){}^{t+\gamma\Delta t}\dot{\mathbf{U}}\Big] \end{aligned} \tag{11} $$

and substituting from Eq. (11) into Eq. (10) we have

$$ \begin{aligned} {}^{t+\Delta t}\ddot{\mathbf{U}} = &\frac{1}{(\beta_2(1-\gamma)\Delta t)^2}\left({}^{t+\Delta t}\mathbf{U} - {}^{t}\mathbf{U}\right) \\ &-\frac{\gamma(1-\beta_1)+\beta_2(1-\gamma)}{(\beta_2(1-\gamma))^2\Delta t}{}^{t}\dot{\mathbf{U}} \\ &-\frac{\gamma\beta_1+(1-\beta_2)(1-\gamma)}{(\beta_2(1-\gamma))^2\Delta t}{}^{t+\gamma\Delta t}\dot{\mathbf{U}} - \frac{\gamma(1-\beta_1)}{\beta_2(1-\gamma)}{}^{t}\ddot{\mathbf{U}} \\ &-\frac{\gamma\beta_1+(1-\beta_2)(1-\gamma)}{\beta_2(1-\gamma)}{}^{t+\gamma\Delta t}\ddot{\mathbf{U}} \end{aligned} \tag{12} $$

Rewriting Eq. (11)

$$ \begin{aligned} {}^{t+\Delta t}\dot{\mathbf{U}} = &\frac{1}{\beta_2(1-\gamma)\Delta t}\left({}^{t+\Delta t}\mathbf{U} - {}^{t}\mathbf{U}\right) - \frac{\gamma(1-\beta_1)}{\beta_2(1-\gamma)}{}^{t}\dot{\mathbf{U}} \\ &-\frac{\gamma\beta_1+(1-\beta_2)(1-\gamma)}{\beta_2(1-\gamma)}{}^{t+\gamma\Delta t}\dot{\mathbf{U}} \end{aligned} \tag{13} $$

Then considering the equilibrium equations at time $t+\Delta t$ we obtain

$$ \mathbf{M}\,{}^{t+\Delta t}\ddot{\mathbf{U}} + \mathbf{C}\,{}^{t+\Delta t}\dot{\mathbf{U}} + \mathbf{K}\,{}^{t+\Delta t}\mathbf{U} = {}^{t+\Delta t}\mathbf{R} \tag{14} $$

and

$$ \widehat{\mathbf{K}}_2\,{}^{t+\Delta t}\mathbf{U} = {}^{t+\Delta t}\widehat{\mathbf{R}}_2 \tag{15} $$

where

$$ \widehat{\mathbf{K}}_2 = \frac{1}{(\beta_2(1-\gamma)\Delta t)^2}\mathbf{M} + \frac{1}{\beta_2(1-\gamma)\Delta t}\mathbf{C} + \mathbf{K} \tag{16} $$

$$ \begin{aligned} {}^{t+\Delta t}\widehat{\mathbf{R}}_2 = {}^{t+\Delta t}\mathbf{R} + \mathbf{M}\Bigg(&\frac{1}{(\beta_2(1-\gamma)\Delta t)^2}{}^{t}\mathbf{U} + \frac{\gamma(1-\beta_1)+\beta_2(1-\gamma)}{(\beta_2(1-\gamma))^2\Delta t}{}^{t}\dot{\mathbf{U}} \\ &+\frac{\gamma\beta_1+(1-\beta_2)(1-\gamma)}{(\beta_2(1-\gamma))^2\Delta t}{}^{t+\gamma\Delta t}\dot{\mathbf{U}} + \frac{\gamma(1-\beta_1)}{\beta_2(1-\gamma)}{}^{t}\ddot{\mathbf{U}} \\ &+\frac{\gamma\beta_1+(1-\beta_2)(1-\gamma)}{\beta_2(1-\gamma)}{}^{t+\gamma\Delta t}\ddot{\mathbf{U}}\Bigg) \\ +\mathbf{C}\Bigg(&\frac{1}{\beta_2(1-\gamma)\Delta t}{}^{t}\mathbf{U} + \frac{\gamma(1-\beta_1)}{\beta_2(1-\gamma)}{}^{t}\dot{\mathbf{U}} + \frac{\gamma\beta_1+(1-\beta_2)(1-\gamma)}{\beta_2(1-\gamma)}{}^{t+\gamma\Delta t}\dot{\mathbf{U}}\Bigg) \end{aligned} \tag{17} $$

The above equations are used recursively, in a standard manner to solve for the discrete solution of the governing finite element equations at the distinct time points Δt apart.

Since we introduced the parameters β_1 and β_2 into the Bathe scheme we refer to the method as the β_1/β_2- Bathe method. We will in this paper focus on the β_1/β_2- Bathe method using always $\gamma=0.5$.

As mentioned already, for $\beta_1=\beta_2=0.5$ the method corresponds to the trapezoidal rule for the first and second half-steps, and for $\beta_1=1/3$ and $\beta_2=2/3$ the procedure is the standard Bathe method using in that method $\gamma=0.5$.

Considering the computational efficiency of the scheme, we notice that two factorizations are needed (Eqs. (5) and (15)) and then two forward-reductions and back-substitutions of vectors are required for each step. However, if $\beta_1=\beta_2=0.5$ with $\gamma=0.5$, the coefficient matrices are the same and only one factorization is needed as in the standard Bathe method with $\gamma=2-\sqrt{2}$ [15].

Furthermore, using the values $\beta_1=1+\frac{1}{2\gamma(\gamma-2)}$ and $\beta_2=\frac{1}{2-\gamma}$, the standard Bathe method is obtained for all γ, see Appendix A. It then turns out that using the β_1/β_2-Bathe method with $\gamma=\frac{2\beta_2}{1+2\beta_2}$ and any

β_1, the coefficient matrices are the same and only one factorization is needed.

3. Stability and accuracy of the β_1/β_2-Bathe scheme

We focus in this section on the stability and accuracy of the solution scheme given above using $\gamma = 0.5$.

3.1. The spectral radius, amplitude decay and period elongation

To analyze the stability, the following recursive formula can be used:

$$\begin{bmatrix} {}^{t+\Delta t}u \\ {}^{t+\Delta t}\dot{u} \\ {}^{t+\Delta t}\ddot{u} \end{bmatrix} = \mathbf{A} \begin{bmatrix} {}^{t}u \\ {}^{t}\dot{u} \\ {}^{t}\ddot{u} \end{bmatrix} + \mathbf{L}_a^{t+\gamma\Delta t} r + \mathbf{L}^{t+\Delta t} r \tag{18}$$

where $\mathbf{A}$, $\mathbf{L}_a$ and $\mathbf{L}$ denote the approximation and load operators, respectively and are given in Appendix B. As seen in the appendix, Eq. (18) contains $\omega, \xi, \beta_1, \beta_2, \gamma$ and Δt, which denote the free vibration frequency, the damping ratio, the parameters, and the time step, respectively.

To investigate the stability of the scheme, the spectral radius $\rho(\mathbf{A})$ with $\xi = 0$ is considered.

The analysis shows that for two regions $\beta_1 \in [1/3, 1/2]$ with $\beta_2 = 1 - \beta_1$ and $\beta_1 \in [1/3, \infty)$, with $\beta_2 = 2\beta_1$ the method is unconditionally stable and gives good accuracy. Hence we focus on the method with the parameters β_1 and β_2 in these regions.

Comparing the performance of the scheme with the standard Bathe method, we see that in the first region $(\beta_1 \in [1/3, 1/2], \quad \beta_2 = 1 - \beta_1)$, as β_1 is increased from 1/3 (the standard Bathe method) to 0.5, the spectral radius $\rho(\mathbf{A})$ increases, and the amplitude decay and period elongation decrease, with of course no amplitude decay at $\beta_1 = \beta_2 = 1/2$, and $\rho(\mathbf{A}) = 1$, see Figs. 1–3.

Further, we see that in the second region $(\beta_1 \in [1/3, \infty), \quad \beta_2 = 2\beta_1)$, as β_1 is increased from 1/3, the spectral radius $\rho(\mathbf{A})$ decreases, and the amplitude decay and period elongation increase (when compared with the values of the standard Bathe method with $\gamma = 0.5$).

As shown in Figs. 1–3, the spectral radii, amplitude decays and period elongations vary smoothly in all cases as the parameters β_1 and β_2 are changed. This is an important characteristic of the scheme. The curves also show that using the values $\beta_1 = 1/3$ and $\beta_2 = 2/3$ (the standard Bathe scheme) is very reasonable (and in some sense optimal) considering the amplitude decay and period elongation but in particular the spectral radius. Fig. 1 shows that

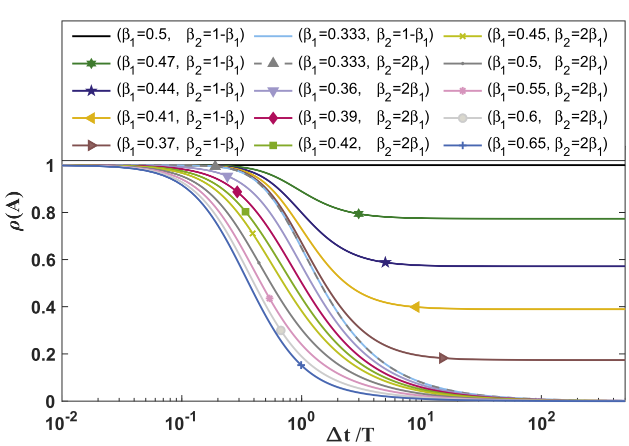

Fig. 1. Spectral radius of the β_1/β_2-Bathe method for various β_1 and β_2 values.

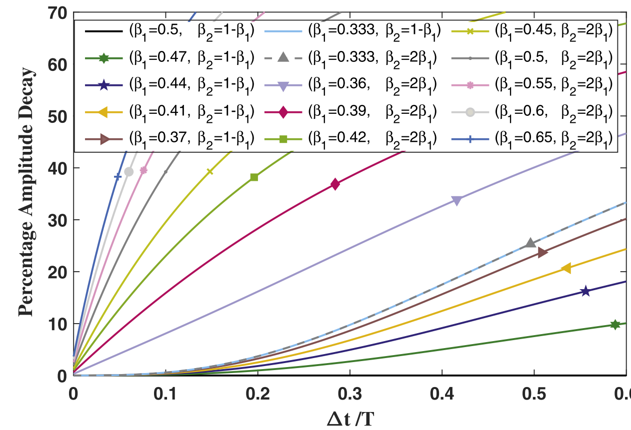

Fig. 2. Percentage amplitude decay of the β_1/β_2-Bathe method.

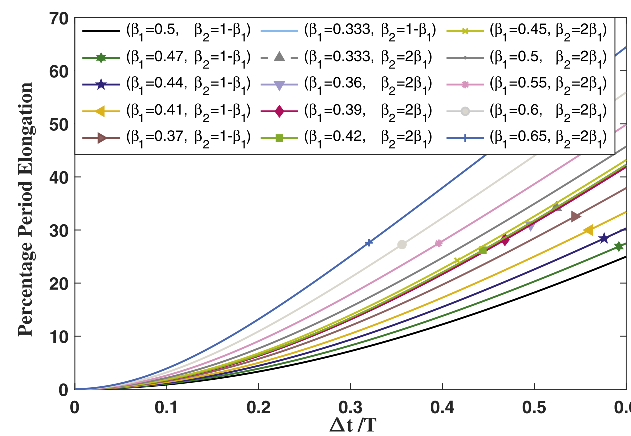

Fig. 3. Percentage period elongations of the β_1/β_2-Bathe method.

using just a small increase for β_1 from 1/3 to 0.37 with $\beta_2 = 0.63$, results into $\rho(\mathbf{A}) \neq 0.0$ for large values of $\Delta t/T$. Hence the higher (spurious) modes will not be eliminated as rapidly.

3.2. The β_1/β_2- Bathe method versus the standard Bathe method

Noh and Bathe investigated the performance of the standard Bathe method in Ref. [11] when γ is changed. They showed that as γ tends to 0 or 1, as expected, the curves of amplitude decay and period elongation of the Bathe method approach those of the trapezoidal rule. We compare in Figs. 4–6, the current scheme with the results given in Ref. [11]. The figures show that in both approaches the spectral radius can be increased or decreased for values of $\Delta t/T$, of course always $\rho(\mathbf{A}) \leqslant 1.0$, and the amplitude decay can be made very large. As expected, using $\gamma = 0.99$ in the standard Bathe method gives in essence the performance of the trapezoidal rule for Δt, but in the Bathe method computations for the two sub-steps are used. This is clearly inefficient. On the other hand, with $\beta_1 = \beta_2 = 1/2$ and $\gamma = 0.5$, we use the trapezoidal rule for each of the sub-steps and obtain more accuracy per time step. Hence this approach is clearly preferable.

3.3. Analysis of possible overshooting

Overshooting is defined as predicting in the first time steps a larger response than given by the exact response when these steps are large with respect to the smallest period in the finite element

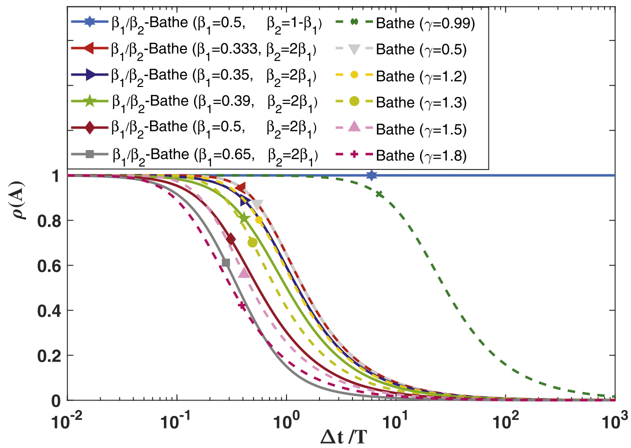

Fig. 4. Spectral radii of β_1/β_2-Bathe method and standard Bathe method.

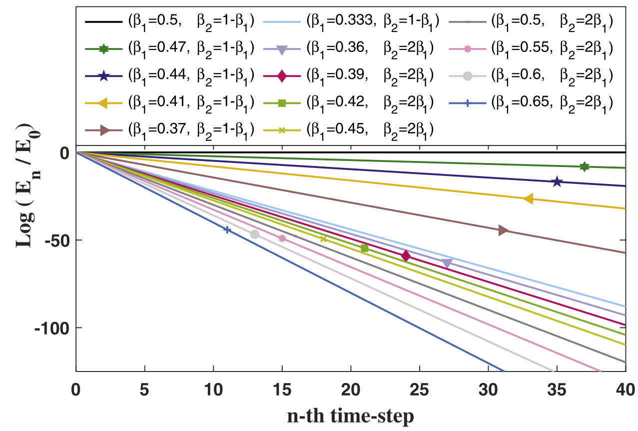

Fig. 7. Energy response of β_1/β_2-Bathe method.

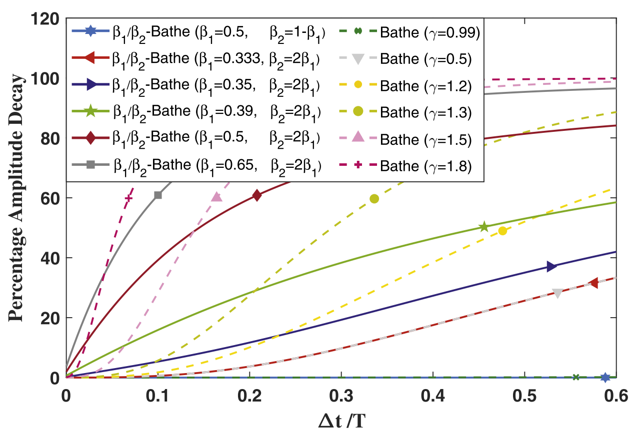

Fig. 5. Percentage amplitude decay of β_1/β_2-Bathe method and standard Bathe method.

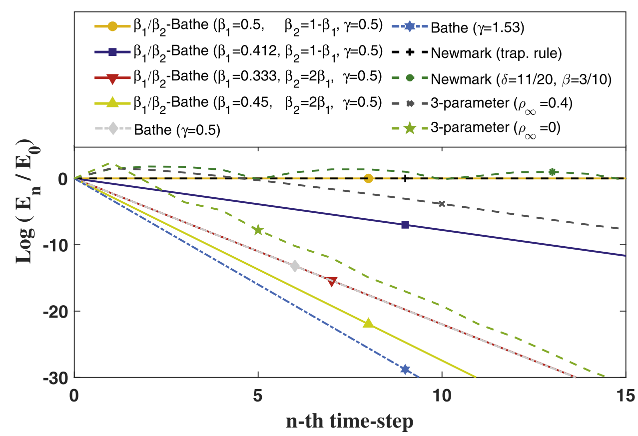

Fig. 8. Comparisons of energy overshoots for various methods.

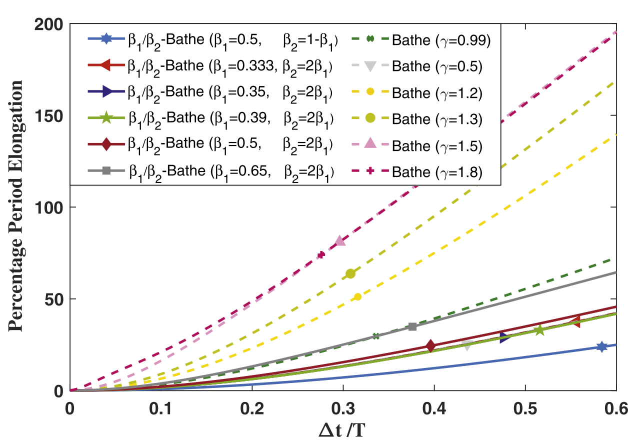

Fig. 6. Percentage period elongation of β_1/β_2-Bathe method and standard Bathe method.

equations. Overshooting, which is unphysical, can be seen in some methods even though these schemes are unconditionally stable.

To measure a possible overshooting, we consider the response of a single degree of freedom system and calculate the energy norm

$$E_n = \|(u_n, \dot{u}_n)\|_E^2 = \left((\dot{u}_n)^2 + \omega^2 (u_n)^2\right)/2 \tag{19}$$

where u_n and $\dot{u}_n$ are the predicted displacement and velocity at time $n\Delta t$. We use the initial conditions ${}^0u = 1.0$, ${}^0\dot{u} = 0.0$ and use $\left(\frac{\Delta t}{T}\right) = 10$.

The predicted normalized energy at the time step n is

$$\frac{E_n}{E_0} = \frac{(\dot{u}_n)^2 + \omega^2 (u_n)^2}{(\dot{u}_0)^2 + \omega^2 (u_0)^2} \tag{20}$$

We aim to have $\log(E_n/E_0) < 0$, as seen for the Bathe schemes, see Fig. 7.

We compare in Fig. 8, the overshooting of some methods and see that the 3-parameter and Newmark methods show overshooting for the parameters used.

4. Illustrative example solutions

We give in this section some solutions that illustrate the properties of the solution schemes.

4.1. Single degree of freedom system

We consider the free vibration solution of a standard single degree of freedom system without physical damping

$$\ddot{u}(t) + 4u(t) = 0, \quad u(0) = 1, \quad \dot{u}(0) = 0 \tag{21}$$

This is a very simple problem to solve and we include it only to highlight some of the observations made already. We use the time

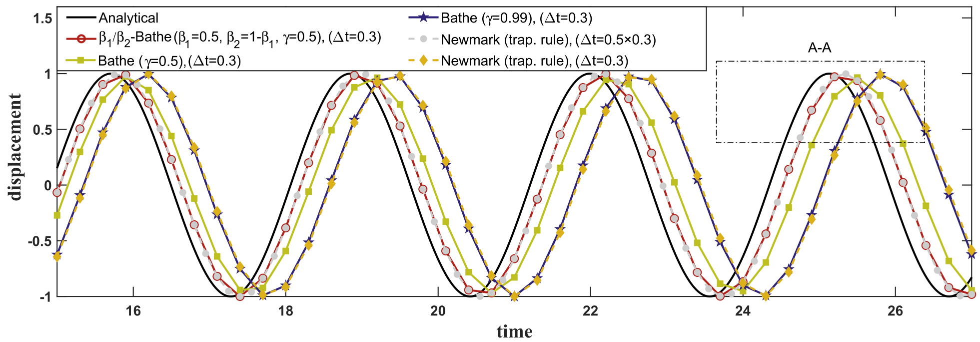

Fig. 9. Displacement response of single degree of freedom system.

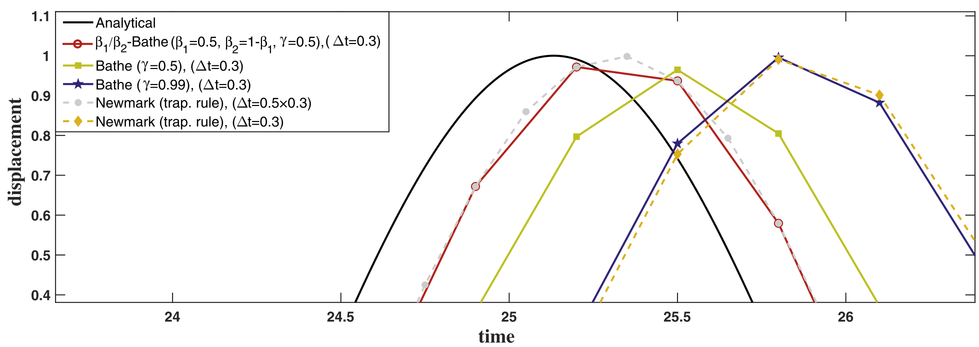

Fig. 10. Section A-A of Fig. 9.

step $\Delta t = 0.3$ and compare the analytical solution with the numerical solutions obtained when using the β_1/β_2-Bathe method, the trapezoidal rule and the standard Bathe method.

Figs. 9 and 10 show that, as expected, the solution using the trapezoidal rule with half the time step size is equal to the solution using the β_1/β_2-Bathe method with β_1 = 0.5 and β_2 = 0.5. Of course, the solution expense is about the same (see Eqs. (1)–(17)). We also see that the solution using the standard Bathe method with $\gamma = 0.99$ is close to the response obtained using the trapezoidal rule with $\Delta t = 0.3$ (but of course the solution with the Bathe method is twice as expensive), and the prediction using the standard Bathe method with $\gamma = 0.5$ is more accurate than when using $\gamma = 0.99$.

Hence these results show that using the β_1/β_2-Bathe method to adjust the numerical dissipation can be more effective than changing the value of γ in the standard scheme.

4.2. A clamped-free bar excited by end load

In this example we consider a clamped bar excited by a step end load (see Fig. 11) [16]. For a model consisting of N finite elements of equal length $\ell = L/N$ the mass and stiffness matrices are

$$
\mathbf{K} = \frac{EA}{\ell}\begin{bmatrix} 2 & -1 & & & & 0 \\ -1 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 2 & -1 \\ 0 & & & & -1 & 1 \end{bmatrix}_{N\times N},
\qquad
\mathbf{M} = \frac{\rho A \ell}{2}\begin{bmatrix} 2 & & & & & 0 \\ & 2 & & & & \\ & & 2 & & & \\ & & & \ddots & & \\ & & & & 2 & \\ 0 & & & & & 1 \end{bmatrix}_{N\times N} \tag{22}
$$

The material and geometrical properties are $E = 30 \times 10^6$, $\rho = 0.00073$, $A = 1$, and $L = 200$. We assume that no physical damping is present. The applied end load is F(t)=10,000. In our solution we use N = 1000.

Figs. 12–22 give the predicted response using the 3-parameter method, the standard Bathe method (in which always the

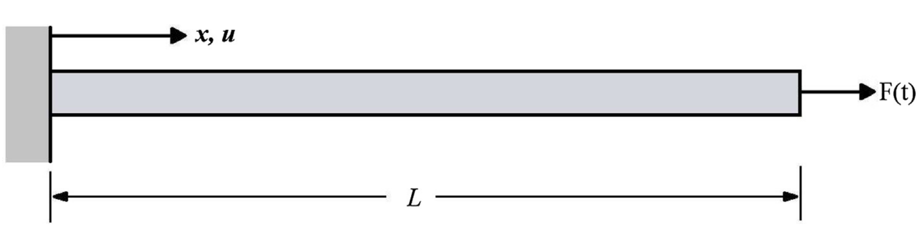

Fig. 11. A clamped–free bar excited by end load.

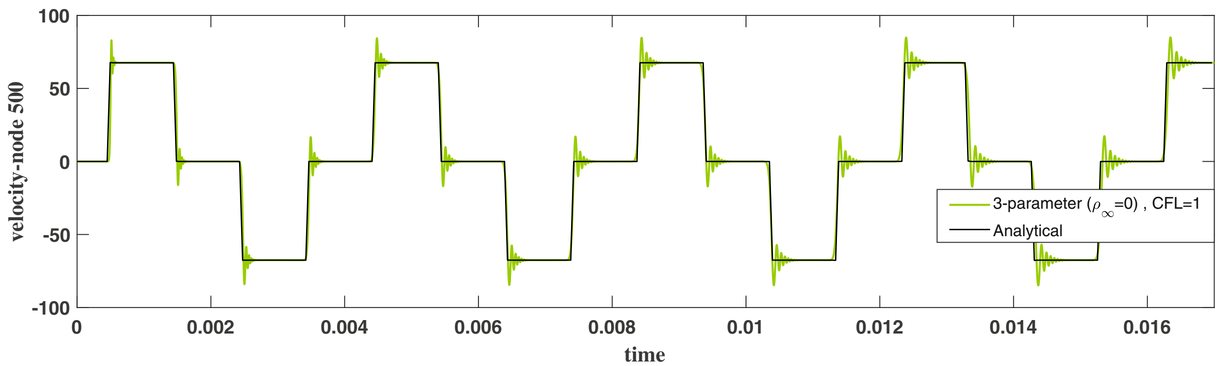

Fig. 12. Velocity of node 500, 3-parameter method.

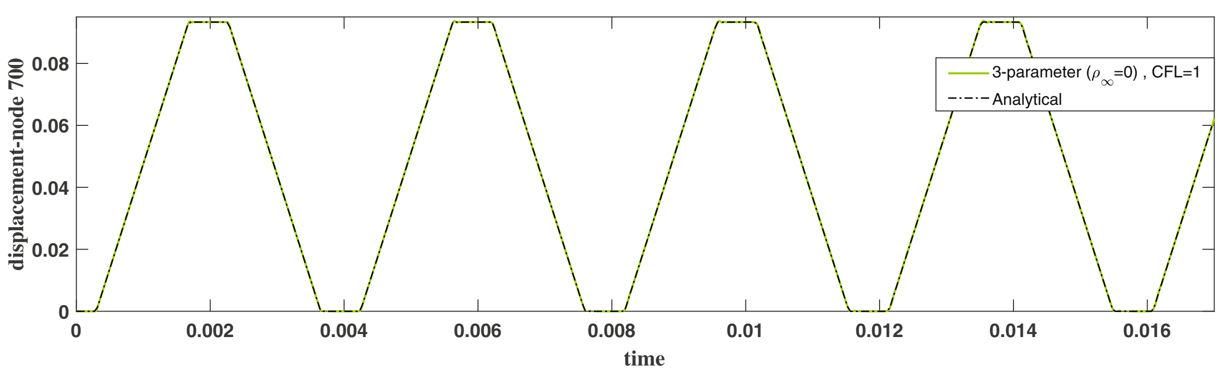

Fig. 13. Displacement of node 700, 3-parameter method.

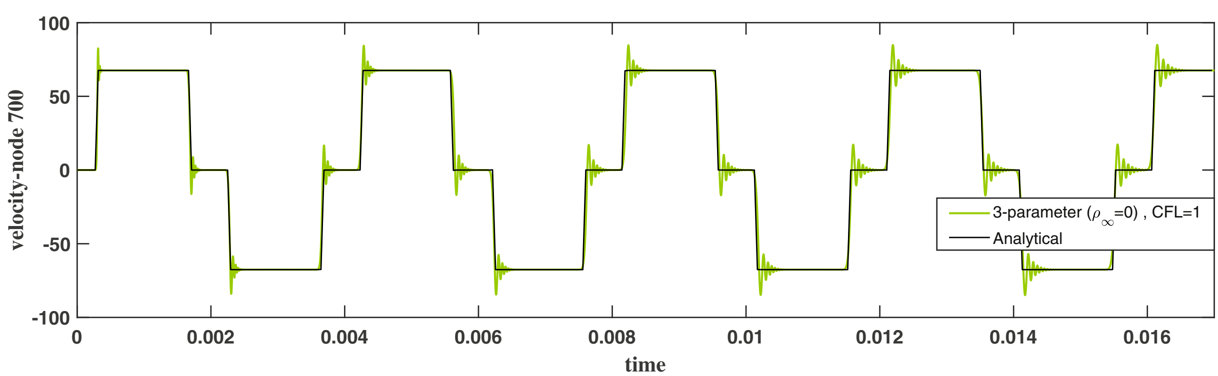

Fig. 14. Velocity of node 700, 3-parameter method.

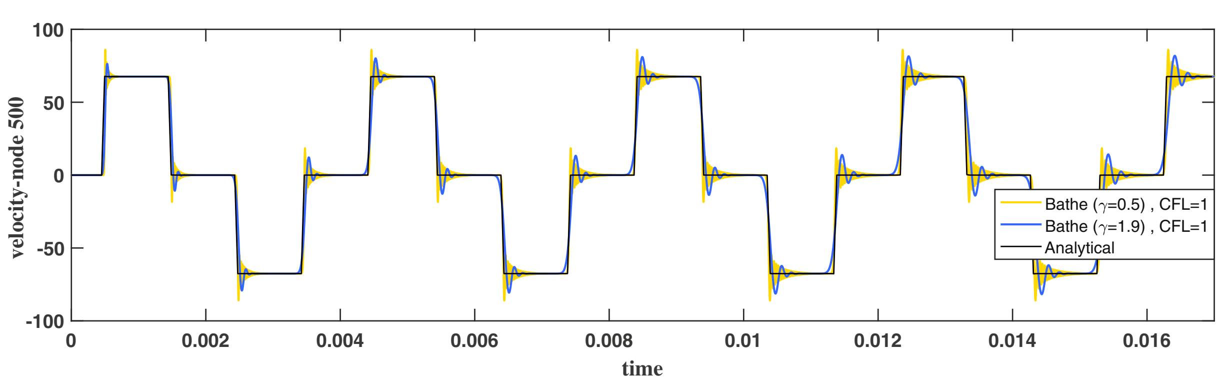

Fig. 15. Velocity of node 500, standard Bathe method.

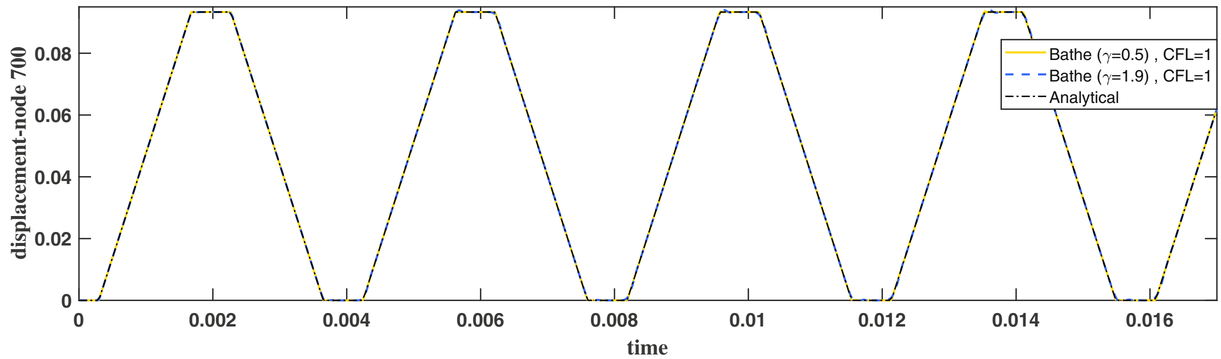

Fig. 16. Displacement of node 700, standard Bathe method.

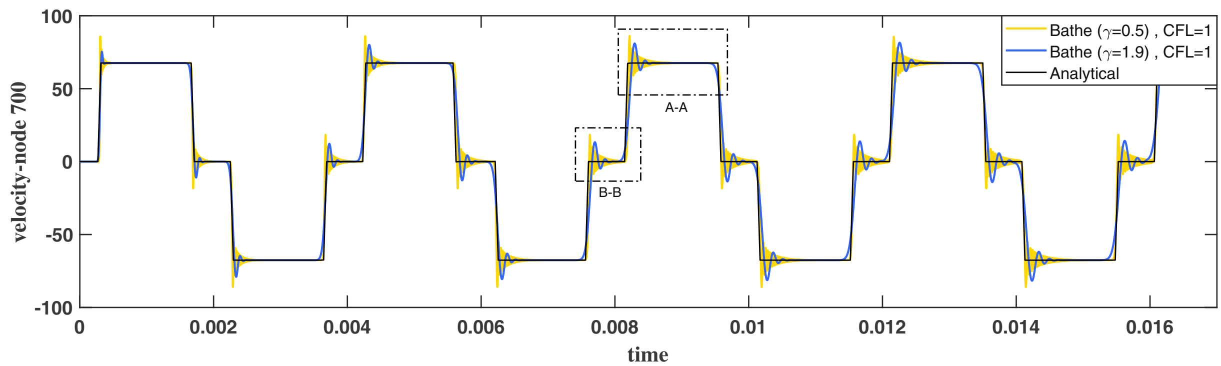

Fig. 17. Velocity of node 700, standard Bathe method.

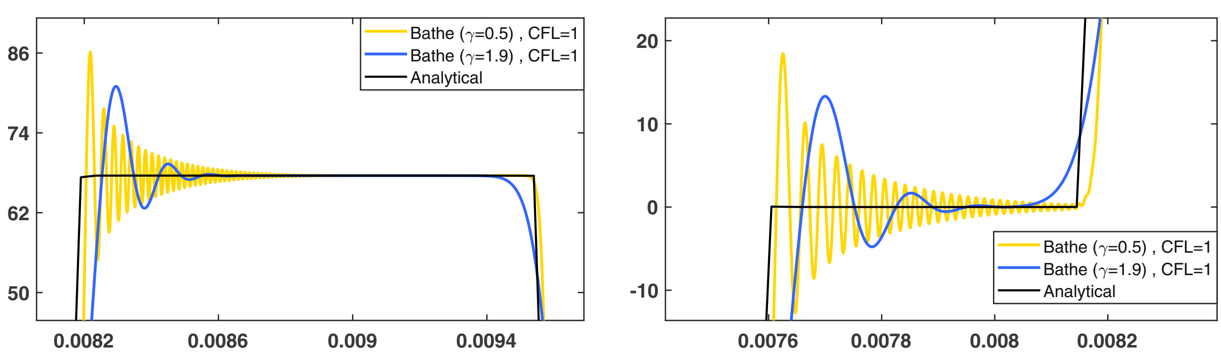

Fig. 18. Sections AA and section BB of Fig. 17.

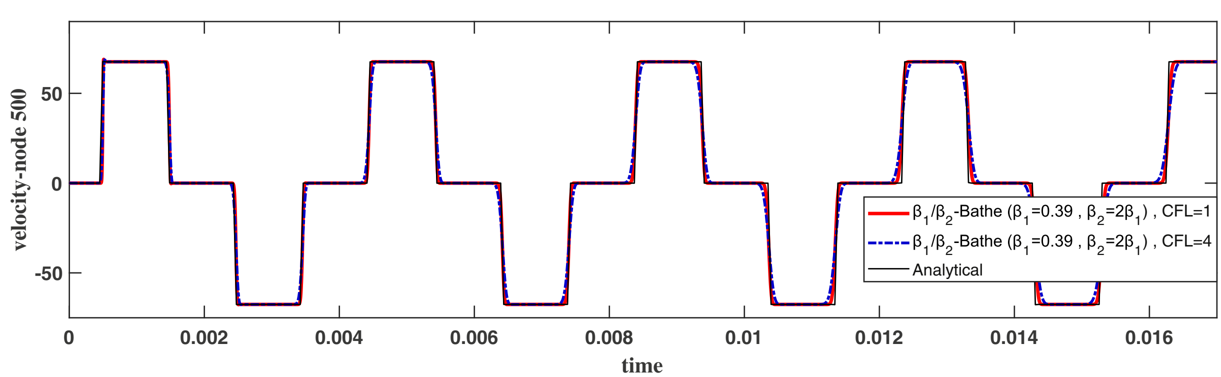

Fig. 19. Velocity of node 500, β_1/β_2-Bathe method.

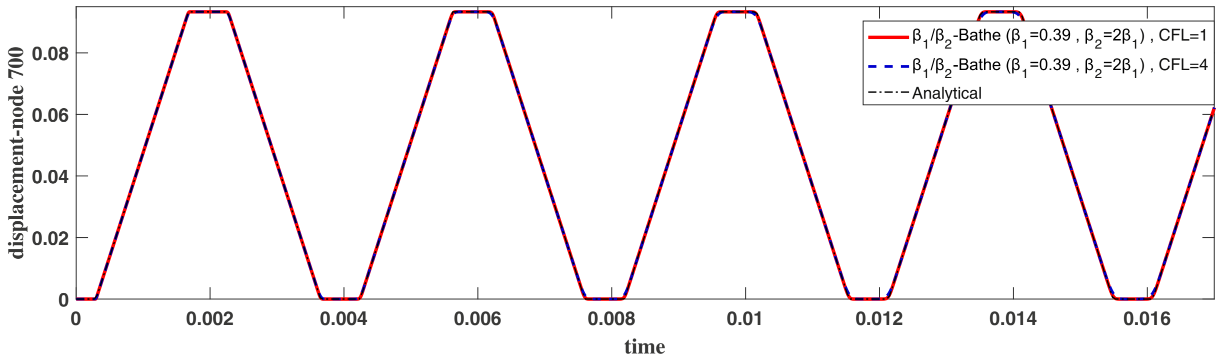

Fig. 20. Displacement of node 700, β_1/β_2-Bathe method.

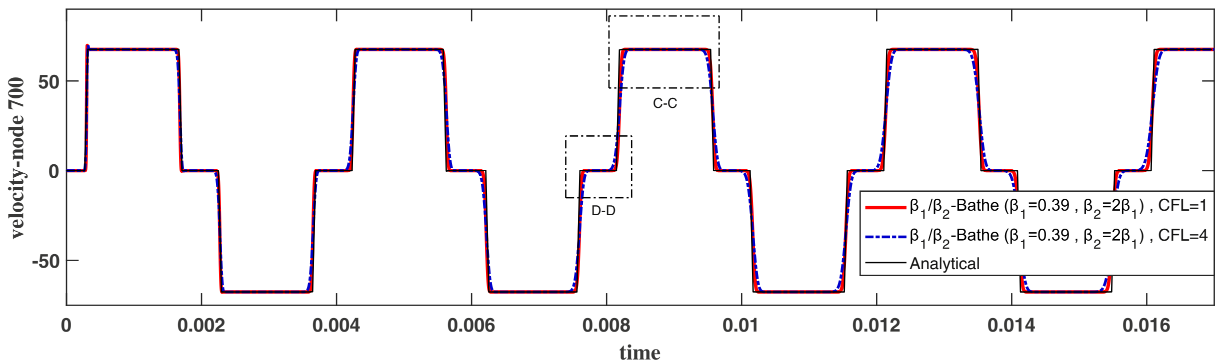

Fig. 21. Velocity of node 700, β_1/β_2-Bathe method.

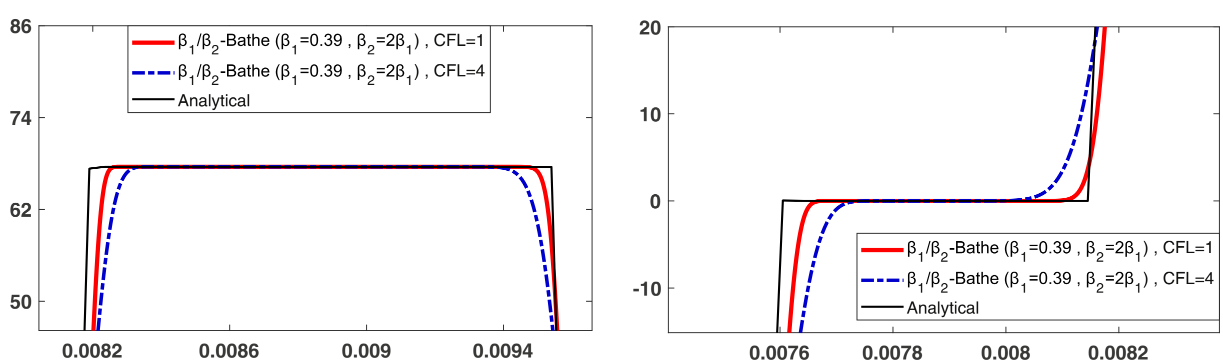

Fig. 22. Section CC and section DD of Fig. 21.

trapezoidal rule is used for the first sub-step), and the β_1/β_2-Bathe method (always using $\gamma = 0.5$). The values of the other parameters used are given in the figures. The time step employed to obtain the results is

$$\Delta t = \left(9.88 \times 10^{-7}\right) \times CFL$$

As previously determined, the standard Bathe method with $\gamma = 0.5$ can give the most accurate response in finite element solutions when CFL = 1.0 [17,18].

In each case we give the predicted velocity at node 500 (at x = 100) and the predicted displacement and velocity at node 700 (at x = 140). The results show that the displacement at node 700 is accurately predicted using any of the methods but the velocity predictions contain some oscillations. The velocity results obtained using the β_1/β_2-Bathe method with the values of the parameters given are clearly more accurate, indeed remarkably accurate for the use of an implicit integration scheme.

However, it should be noted that to obtain these accurate results requires the selection of "good" values for the β_1 and β_2 parameters and this may require some numerical experimentation. In this problem solution, the choice of parameter values should yield no oscillations in the response (as for example seen for the velocity predictions using the standard Bathe method) which helps to choose effective parameter values.

5. Concluding remarks

We revisited in this paper the Bathe implicit time integration procedure with the objective to be able to introduce a desired numerical dissipation. In general, the standard Bathe scheme is used with three parameter values that are set and effective in many applications, and in industry will likely not be changed. These parameters are $\delta = 0.5$, $\alpha = 1/4$ in the Newmark method to have the trapezoidal rule in the first sub-step, and the time step splitting ratio $\gamma = 0.5$ to split the complete time step into two sub-steps. The effect of using other values for γ to introduce different numerical dissipation than used in the standard scheme was previously studied [11].

In this paper we used for the second sub-step the usual Newmark approximation (given as δ in Ref. [15]) twice for each sub-step and thus introduced the two parameters β_1 and β_2 to obtain the β_1/β_2-Bathe method. While the sub-steps can be of different size, we focused in our analysis and use of the method on equal size sub-steps, i.e. $\gamma = 0.5$. With $\beta_1 = 1/3$ and $\beta_2 = 2/3$, the method is then the standard Bathe method and with $\beta_1 = 0.5$ and $\beta_2 = 0.5$ the trapezoidal rule is also used for the second sub-step. Of course, with the use of the trapezoidal rule for each of the two sub-steps, no amplitude decay occurs over the complete step and the numerical errors are as in the single step trapezoidal rule applied to each sub-step.

We observed that using $\beta_1 = 1/3$ and $\beta_2 = 2/3$ (giving the standard Bathe method) is reasonable and in some sense optimal because for these values the spectral radius stays equal to 1.0 as $\Delta t/T$ increases until the largest value of $\Delta t/T$ is reached for which thereafter the spectral radius rapidly decreases to zero. This property results into best overall accuracy in the integration of the lowest modes *and* a rapid suppression of spurious response in the higher modes of the finite element system.

However, with the approach of using the β_1, β_2 parameters, if so desired, an analyst can change the values of the parameters and can introduce a desired numerical amplitude decay with an associated period elongation. Using always $\gamma = 0.5$, more or less numerical dissipation than employed in the standard Bathe method can be introduced. An important observation is that the amount of numerical dissipation changes smoothly from zero to very large as the parameters are changed. The corresponding period elongation also changes smoothly. As shown in the paper, this approach can be more effective than changing γ in the standard Bathe method.

We studied the use of the β_1, β_2 parameters and showed that employing the β_1/β_2-Bathe method can yield accurate solutions of dynamic problems that can be difficult to reach. Hence the β_1/β_2-Bathe method can be of value when an analyst desires to change the dissipation in the standard Bathe scheme. However, to obtain the most accurate response for a given problem solution will probably require some numerical experimentation.

Since we have only used $\gamma = 0.5$ with nonstandard values for β_1 and β_2, a further study could be performed to see how a change of γ will affect the response when the parameters β_1 and β_2 take on different values.

Appendix A. Proof that the values $\beta_1 = 1 + \frac{1}{2\gamma(\gamma-2)}$ and $\beta_2 = \frac{1}{2-\gamma}$ give the standard Bathe scheme

Using $\beta_1 = 1 + \frac{1}{2\gamma(\gamma-2)}$ and $\beta_2 = \frac{1}{2-\gamma}$ in Eqs. (10) and (11), we obtain

$$ {}^{t+\Delta t}\ddot{\mathbf{U}} = \frac{(2-\gamma)}{(1-\gamma)\Delta t}\left({}^{t+\Delta t}\dot{\mathbf{U}} - {}^{t}\dot{\mathbf{U}}\right) + \frac{1}{2(\gamma-1)}\left({}^{t}\ddot{\mathbf{U}} + {}^{t+\gamma\Delta t}\ddot{\mathbf{U}}\right) \tag{23} $$

$$ {}^{t+\Delta t}\dot{\mathbf{U}} = \frac{(2-\gamma)}{(1-\gamma)\Delta t}\left({}^{t+\Delta t}\mathbf{U} - {}^{t}\mathbf{U}\right) + \frac{1}{2(\gamma-1)}\left({}^{t}\dot{\mathbf{U}} + {}^{t+\gamma\Delta t}\dot{\mathbf{U}}\right) \tag{24} $$

and Eqs. (1) and (2) are

$$ \left({}^{t}\ddot{\mathbf{U}} + {}^{t+\gamma\Delta t}\ddot{\mathbf{U}}\right) = \left(\frac{2}{\gamma\Delta t}\right)\left({}^{t+\gamma\Delta t}\dot{\mathbf{U}} - {}^{t}\dot{\mathbf{U}}\right) \tag{25} $$

$$ \left({}^{t}\dot{\mathbf{U}} + {}^{t+\gamma\Delta t}\dot{\mathbf{U}}\right) = \left(\frac{2}{\gamma\Delta t}\right)\left({}^{t+\gamma\Delta t}\mathbf{U} - {}^{t}\mathbf{U}\right) \tag{26} $$

Using Eq. (25) in Eq. (23) and Eq. (26) in Eq. (24), we have

$$ {}^{t+\Delta t}\ddot{\mathbf{U}} = \frac{(1-\gamma)}{\gamma\Delta t}{}^{t}\dot{\mathbf{U}} - \frac{1}{(1-\gamma)\gamma\Delta t}{}^{t+\gamma\Delta t}\dot{\mathbf{U}} + \frac{(2-\gamma)}{(1-\gamma)\Delta t}{}^{t+\Delta t}\dot{\mathbf{U}} \tag{27} $$

$$ {}^{t+\Delta t}\dot{\mathbf{U}} = \frac{(1-\gamma)}{\gamma\Delta t}{}^{t}\mathbf{U} - \frac{1}{(1-\gamma)\gamma\Delta t}{}^{t+\gamma\Delta t}\mathbf{U} + \frac{(2-\gamma)}{(1-\gamma)\Delta t}{}^{t+\Delta t}\mathbf{U} \tag{28} $$

These are the equations used in the second sub-step in the standard Bathe method.

Appendix B. The approximation and load operators of the β_1/β_2-Bathe method

$$ \begin{bmatrix} {}^{t+\Delta t}u \\ {}^{t+\Delta t}\dot{u} \\ {}^{t+\Delta t}\ddot{u} \end{bmatrix} = \mathbf{A}\begin{bmatrix} {}^{t}u \\ {}^{t}\dot{u} \\ {}^{t}\ddot{u} \end{bmatrix} + \mathbf{L}_a^{t+\gamma\Delta t}r + \mathbf{L}^{t+\Delta t}r \tag{29} $$

where

$$ \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{L}_a = \begin{Bmatrix} l_1 \\ l_2 \\ l_3 \end{Bmatrix}, \quad \mathbf{L}\begin{Bmatrix} l_4 \\ l_5 \\ l_6 \end{Bmatrix} \tag{30} $$

$$ \begin{aligned}
a_0 &= \frac{(\gamma\Delta t)}{2}, \\
a_1 &= a_0^2, \\
a_2 &= 2a_0 \\
a_3 &= \frac{1}{\beta_2(1-\gamma)\Delta t}, \\
a_4 &= a_3^2, \\
a_5 &= \frac{\gamma(1-\beta_1) + \beta_2(1-\gamma)}{(\beta_2(1-\gamma))^2\Delta t}, \\
a_6 &= \frac{\gamma\beta_1 + (1-\beta_2)(1-\gamma)}{(\beta_2(1-\gamma))^2\Delta t}, \\
a_7 &= \frac{\gamma(1-\beta_1)}{\beta_2(1-\gamma)}, \\
a_8 &= \frac{\gamma\beta_1 + (1-\beta_2)(1-\gamma)}{\beta_2(1-\gamma)} \\
D &= \left(1 + 2\xi\omega a_0 + \omega^2 a_1\right)^{-1}, \\
D_1 &= \left(a_4 + 2\xi\omega a_3 + \omega^2\right)^{-1} \\
b_1 &= -D\omega^2, \\
b_2 &= D\left(-2\xi\omega - \omega^2 a_2\right), \\
b_3 &= D\left(-2\xi\omega a_0 - \omega^2 a_1\right) \\
b_4 &= a_0 b_1 \\
b_5 &= 1 + a_0 b_2 \\
b_6 &= a_0 + a_0 b_3 \\
a_{11} &= D_1(a_4 + 2\xi\omega a_3 + b_4(a_6 + 2\xi\omega a_8) + b_1 a_8) \\
a_{12} &= D_1(a_5 + 2\xi\omega a_7 + b_5(a_6 + 2\xi\omega a_8) + b_2 a_8) \\
a_{13} &= D_1(a_7 + b_6(a_6 + 2\xi\omega a_8) + b_3 a_8), \\
a_{21} &= a_{11}a_3 - a_3 - b_4 a_8 \\
a_{22} &= a_{12}a_3 - a_7 - b_5 a_8 \\
a_{23} &= a_{13}a_3 - b_6 a_8, \\
a_{31} &= a_{21}a_3 - b_1 a_8 \\
a_{32} &= a_{22}a_3 - a_3 - b_2 a_8 \\
a_{33} &= a_{23}a_3 - a_7 - b_3 a_8 \\
l_1 &= D_1 D(a_0(a_6 + 2\xi\omega a_8) + a_8) \\
l_2 &= l_1 a_3 - D a_0 a_8 \\
l_3 &= l_2 a_3 - D a_8 \\
l_4 &= D_1 \\
l_5 &= a_3 D_1 \\
l_6 &= a_3^2 D_1
\end{aligned} $$

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