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A HYPERELASTIC-BASED LARGE STRAIN ELASTO-PLASTIC CONSTITUTIVE FORMULATION WITH COMBINED ISOTROPIC–KINEMATIC HARDENING USING THE LOGARITHMIC STRESS AND STRAIN MEASURES

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SUMMARY

This paper addresses the formulation of a set of constitutive equations for finite deformation metal plasticity. The combined isotropic–kinematic hardening model of the infinitesimal theory of plasticity is extended to the large strain range on the basis of three main assumptions: (i) the formulation is hyperelastic based, (ii) the stress–strain law preserves the elastic constants of the infinitesimal theory but is written in terms of the Hencky strain tensor and its elastic work conjugate stress tensor, and (iii) the multiplicative decomposition of the deformation gradient is adopted. Since no stress rates are present, the formulation is, of course, numerically objective in the time integration. It is shown that the model gives adequate physical behaviour, and comparison is made with an equivalent constitutive model based on the additive decomposition of the strain tensor.

INTRODUCTION

Constitutive modelling of elasto-plastic materials in the finite deformation range has received considerable attention over the past years. The availability of powerful computers and efficient finite element techniques has made feasible the solution of large scale finite deformation problems, thus increasing the demand for more realistic and accurate models.

An extension of the infinitesimal theory of plasticity to the large strain range is not straightforward, and a number of alternative formulations, all having identical 'small strain limit', have been proposed. In these formulations basic questions arise about the proper kinematic description of plastic flow, the characterization of the underlying elastic behaviour and the choice of adequate stress and strain measures.

The objective in our development was to arrive at a large strain elasto-plastic constitutive formulation for efficient finite element analysis that is a direct extension of materially-non-linear-only and large displacement/small strain analysis.[1–3] An important feature of the formulations for materially-non-linear-only and large displacement/small strain analysis in References 1–3, in

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abundant use since 1973, is that they are *total* and not rate-type formulations. This leads to physically realistic and computationally efficient solutions of general inelastic response.[3–5]

With the above aim, a natural approach to large strain elasto-plastic analysis is based on the use of a hyperelastic material description with the Hencky strain and the product decomposition of the deformation gradient into elastic and plastic parts. The reasons for such choice can be briefly ennumerated as follows.

The earliest constitutive models for large strain plasticity described plastic flow by the additive decomposition of the strain tensor into elastic and plastic parts, see, for example, Green and Naghdi[6] and Perzyna and Wojno.[7,8] To ensure with such formulation that the plastic formulation is isochoric it is necessary to use the Hencky strain.[9,10] However, when using this strain measure and the corresponding energy conjugate stress measure, the elastic shear modulus becomes, for certain loading conditions, a function of the plastic state, resulting in a decay of the initial shear modulus.[11] Although this decay does not occur in many practical cases, and is not very significant for strain levels below 40 per cent, it clearly is an undesirable phenomenon.

An alternative description of large strain plastic flow is based on the multiplicative decomposition of the deformation gradient into elastic and plastic parts and was introduced by Lee and Liu;[12,13] see also References 14–17. In this case the elastic strain tensor is obtained from the elastic deformation gradient, which ensures that the elastic material moduli remain constant and, using the appropriate flow rule, that the plastic deformation is isochoric.

Large strain plasticity formulations are mostly based on a hypoelastic stress–strain law.[18] However, such formulations lead to non-conservative response predictions in purely elastic cyclic motions; i.e. considering a closed elastic strain path the exact solution using a hypoelastic formulation leads to residual stresses that are clearly non-physical.[19] This effect may be deemed negligible when both the elastic stretches and stretch rates are small, but is clearly always an undesirable phenomenon.

Another consideration regarding hypoelastic material descriptions is that the time integration of the rate quantities leads to numerical errors, even if numerical objectivity is preserved.[20–24] Hence, considering for example an elastic response involving rotations, a rather large number of integration steps is needed for an accurate response prediction. On the other hand, a total formulation based on hyperelasticity allows much larger integration steps and does not lead to non-conservative response predictions in elastic conditions.[3,19]

Based on the advantages of a hyperelastic approach, Simo established a framework for large strain elasto-plastic constitutive equations using the product decomposition,[25,26] and Weber and Anand developed a constitutive model using the Hencky strain for hyperelastic visco-plastic solids with isotropic hardening.[27] The material presented in this paper constitutes a summary and continuation of our earlier research[9,10] and complements the work of Simo, and Weber and Anand, in that a hyperelastic-based description and the product decomposition of the deformation gradient are also employed. However, we focus on a combined isotropic–kinematic hardening model using the Hencky strain tensor, and the evaluation of this model against an equivalent model based on the additive decomposition of the strains.

We consider in this paper the development of the material model and the time integration of the resulting constitutive equations and do not address issues of the solution of the finite element equilibrium equations (such as the derivation of the consistent tangent stiffness matrix).

In the following sections we first present the constitutive equations that we use and then an algorithm for the numerical integration of these equations. This numerical procedure uses the effective-stress-function approach of References 4 and 5. Finally, we show some simple stress–strain response predictions obtained with the model and conclude that the model gives physically realistic response predictions while, of course, being numerically very efficient.

CONSTITUTIVE EQUATIONS

State variables

We consider a class of elasto-plastic constitutive models for rate-independent metal plasticity based on the array (σ, $\mathbf{B}$) of internal variables. The scalar σ, called deformation resistance, has the dimension of stress and represents an average isotropic resistance to macroscopic plastic flow offered by underlying isotropic strengthening mechanisms. The traceless second-order tensor $\mathbf{B}$, called back stress tensor, has the dimension of stress and can be interpreted as an average intensity of the microscopic residual stress which arises from anisotropic microstructural effects.[28]

To characterize the large strain elasto-plastic deformation of a body we use the multiplicative decomposition of the deformation gradient[12,13]

$$\mathbf{X} = \mathbf{X}^{e}\mathbf{X}^{p} \tag{1}$$

where $\mathbf{X}^e$ and $\mathbf{X}^p$ represent respectively the elastic and plastic deformation gradients. Central to this approach is the concept of a relaxed intermediate configuration which for each particle is conceptually obtained by unloading the material neighbourhood of the particle from the current configuration to a state of zero stress in such a way that no inelastic process takes place during the deformation.[12–17] The plastic deformation gradient $\mathbf{X}^p$ corresponds to the deformation from the original to the intermediate configuration. Since plastic deformation is assumed to be incompressible, we have $J^p = \det \mathbf{X}^p = 1$. Having specified $\mathbf{X}^p$, the elastic deformation gradient $\mathbf{X}^e$ is given by $\mathbf{X}^e = \mathbf{X}(\mathbf{X}^p)^{-1}$, and the multiplicative decomposition (1) follows. Consequently, the velocity gradient $\mathbf{L} = \dot{\mathbf{X}}\mathbf{X}^{-1}$ can be written as

$$\mathbf{L} = \mathbf{L}^{e} + \mathbf{L}^{p} \tag{2}$$

where the elastic and plastic velocity gradients are given respectively by $\mathbf{L}^e = \dot{\mathbf{X}}^e(\mathbf{X}^e)^{-1}$ and $\mathbf{L}^p = \mathbf{X}^e\dot{\mathbf{X}}^p(\mathbf{X}^p)^{-1}(\mathbf{X}^e)^{-1}$.

The set of variables that characterize an elasto-plastic process is therefore given by $\{\mathbf{X}, \mathbf{X}^p, \boldsymbol{\tau}, \sigma, \mathbf{B}\}$, where $\boldsymbol{\tau}$ is the Cauchy stress tensor. A constitutive equation for $\boldsymbol{\tau}$ and evolution equations for $\mathbf{X}^p$, σ and $\mathbf{B}$ have to be identified to completely specify the material response. In the formulation of the governing equations for large strain analysis, it is useful to select a set of associated variables that are invariant under rigid body motions. Equations written in terms of these invariant variables satisfy automatically the principle of material frame indifference.

Since $J^p = 1$, it follows from (1) that $J^e = \det \mathbf{X}^e = J > 0$ and the elastic deformation gradient admits the polar decomposition

$$\mathbf{X}^{e} = \mathbf{R}^{e}\mathbf{U}^{e} \tag{3}$$

where $\mathbf{R}^e$ is the elastic rotation tensor and $\mathbf{U}^e$ is the elastic right stretch tensor. We consider the elastic Hencky strain tensor[29]

$$\mathbf{E}^{e} = \ln \mathbf{U}^{e} \tag{4}$$

An invariant stress quantity can now be obtained by imposing elastic work conjugacy with $\mathbf{E}^e$,

$$\bar{\boldsymbol{\tau}} \cdot \dot{\mathbf{E}}^{e} = \boldsymbol{\tau} \cdot \mathbf{D}^{e} \tag{5}$$

where $\mathbf{D}^e = \operatorname{sym} \mathbf{L}^e$ is the elastic velocity strain tensor. When $[\bar{\boldsymbol{\tau}}, \mathbf{E}^e] = \mathbf{0}$, i.e. when the stress and strain tensors commute, the 'rotated' stress tensor $\bar{\boldsymbol{\tau}}$ can be written as

$$\bar{\boldsymbol{\tau}} = (\mathbf{R}^{e})^{\mathrm{T}}\boldsymbol{\tau}\mathbf{R}^{e} \tag{6}$$

Similarly we introduce the 'rotated' back stress $\bar{\mathbf{B}}$,

$$\bar{\mathbf{B}} = (\mathbf{R}^e)^T \mathbf{B} \mathbf{R}^e \tag{7}$$

and the 'modified' plastic velocity gradient

$$\bar{\mathbf{L}}^p = (\mathbf{X}^e)^{-1} \mathbf{L}^p \mathbf{X}^e = \dot{\mathbf{X}}^p (\mathbf{X}^p)^{-1} \tag{8}$$

Quantities $\bar{\tau}$, $\bar{\mathbf{B}}$ and $\bar{\mathbf{L}}^p$ are invariant under rigid body motions, and can be thought of as being associated with the intermediate configuration.

The reduced dissipation inequality

Taking $\{\mathbf{E}^e, \sigma, \bar{\mathbf{B}}\}$ as state variables, the reduced dissipation inequality for isothermal processes can be written as

$$-\dot{\psi} + \bar{\tau} \cdot \dot{\mathbf{E}}^e + (\mathbf{X}^e)^T \tau (\mathbf{X}^e)^{-T} \cdot \bar{\mathbf{L}}^p \geqslant 0 \tag{9}$$

where $\psi(\mathbf{E}^e, \sigma, \bar{\mathbf{B}})$ is the free energy per unit current volume. Assuming that the free energy density can be separated into

$$\psi(\mathbf{E}^e, \sigma, \bar{\mathbf{B}}) = \psi^e(\mathbf{E}^e) + \psi^p(\sigma, \bar{\mathbf{B}}) \tag{10}$$

equation (9) reduces to

$$\left(\bar{\tau} - \frac{\partial \psi^e}{\partial \mathbf{E}^e}\right) \cdot \dot{\mathbf{E}}^e + (\mathbf{X}^e)^T \tau (\mathbf{X}^e)^{-T} \cdot \bar{\mathbf{L}}^p - \dot{\psi}^p \geqslant 0 \tag{11}$$

Assuming that equation (11) has to hold for all motions, the following conditions are derived:[30]

$$\bar{\tau} = \frac{\partial \psi^e}{\partial \mathbf{E}^e} \tag{12}$$

$$D \equiv \mathbf{U}^e \bar{\tau} (\mathbf{U}^e)^{-1} \cdot \bar{\mathbf{L}}^p - \dot{\psi}^p \geqslant 0 \tag{13}$$

where D is called the dissipation function.

Constitutive equation for stress

The stress–strain law is taken to be

$$\bar{\tau} = \mathscr{L}[\mathbf{E}^e] \tag{14}$$

where $\mathscr{L}$ is the fourth-order isotropic elastic moduli tensor. It can be written as

$$\mathscr{L} = 2\mu \mathbf{I} + (\kappa - \tfrac{2}{3}\mu) \mathbf{1} \otimes \mathbf{1} \tag{15}$$

where $\mathbf{1}$ and $\mathbf{I}$ are respectively the second- and fourth-order identity tensors, μ is the shear modulus and κ is the bulk modulus. This choice of hyperelastic stress–strain law (using therefore the total elastic strain) has the advantage of providing an excellent description of moderate elastic strains yet depending only on the material constants μ and κ.

Since the stress–strain law (14) is isotropic, the dissipation function (13) takes the form

$$D(\bar{\tau}; \bar{\mathbf{D}}^p, \dot{\psi}^p) = \bar{\tau} \cdot \bar{\mathbf{D}}^p - \dot{\psi}^p \tag{16}$$

where $\bar{\mathbf{D}}^p = \operatorname{sym} \bar{\mathbf{L}}^p$. We use this function to derive the evolution equation for $\mathbf{X}^p$.

Elastic domain

Let S be the 'effective stress tensor', given in terms of the Cauchy and back stress tensors by

$$\mathbf{S} = \boldsymbol{\tau}' - \mathbf{B} \tag{17}$$

where $\boldsymbol{\tau}' = \operatorname{dev} \boldsymbol{\tau}$ indicates the deviatoric part, and denote by s the effective stress

$$s = \sqrt{\tfrac{3}{2}\mathbf{S}\cdot\mathbf{S}} \tag{18}$$

The classical 'elastic domain' for combined isotropic–kinematic hardening rate-independent plasticity is given by

$$E_{(\sigma,\mathbf{B})} = \{\boldsymbol{\tau}\colon \Phi(s,\sigma) = s - \sigma < 0\} \tag{19}$$

with the boundary of $E_{(\sigma,\mathbf{B})}$, the 'yield surface', defined by the condition

$$\Phi(s,\sigma) = s - \sigma = 0 \tag{20}$$

In view of (6)–(7) we define the 'rotated' effective stress tensor by

$$\bar{\mathbf{S}} = (\mathbf{R}^{e})^{\mathrm{T}}\mathbf{S}\mathbf{R}^{e} = \bar{\boldsymbol{\tau}}' - \bar{\mathbf{B}} \tag{21}$$

In terms of this definition the effective stress is

$$s = \sqrt{\tfrac{3}{2}\bar{\mathbf{S}}\cdot\bar{\mathbf{S}}} \tag{22}$$

The unit normal $\bar{\mathbf{N}}$ to the yield surface in the rotated stress space is

$$\bar{\mathbf{N}} = \sqrt{\frac{3}{2}}\frac{\bar{\mathbf{S}}}{s} \tag{23}$$

The evolution equation for $\mathbf{X}^{p}$

The evolution equation for the plastic deformation gradient is given by

$$\dot{\mathbf{X}}^{p} = \bar{\mathbf{L}}^{p}\mathbf{X}^{p} \tag{24}$$

The modified plastic stretching tensor $\bar{\mathbf{D}}^{p} = \operatorname{sym}\bar{\mathbf{L}}^{p}$ is specified by the principle of maximum plastic dissipation[31,32] as follows. For fixed plastic variables ($\bar{\mathbf{D}}^{p}$, σ, $\bar{\mathbf{B}}$), the actual stress state $\bar{\boldsymbol{\tau}}$ maximizes the dissipation function $D(\bar{\boldsymbol{\tau}}; \bar{\mathbf{D}}^{p}, \dot{\psi}^{p})$ subject to the constraint $\Phi \leqslant 0$. The first-order necessary conditions for such a maximum are (see for example Reference 33)

$$\frac{\partial D}{\partial \bar{\boldsymbol{\tau}}} - \dot{e}^{p}\frac{\partial \Phi}{\partial \bar{\boldsymbol{\tau}}} = 0 \tag{25}$$

$$\Phi \leqslant 0 \qquad \dot{e}^{p} \geqslant 0 \qquad \dot{e}^{p}\Phi = 0 \tag{26}$$

We also have the consistency condition

$$\dot{e}^{p}\dot{\Phi} = 0 \tag{27}$$

Using the above equations we obtain for the flow rule

$$\bar{\mathbf{D}}^{p} = \sqrt{\tfrac{3}{2}}\,\dot{e}^{p}\bar{\mathbf{N}} \tag{28}$$

In these equations $\dot{e}^{p}$ is the equivalent plastic strain rate

$$\dot{e}^{p} = \sqrt{\tfrac{2}{3}\bar{\mathbf{D}}^{p}\cdot\bar{\mathbf{D}}^{p}} \tag{29}$$

Consistent with the simplifications of the present theory, the modified plastic spin tensor $\bar{\mathbf{W}}^{\mathrm{p}} = \operatorname{skw} \bar{\mathbf{L}}^{\mathrm{p}}$ is taken to be zero.

The evolution equations for σ and $\bar{\mathbf{B}}$

The evolution equations for the modified deformation resistance and the rotated back stress tensor are given by the hardening rules

$$\dot{\sigma} = \beta H \dot{e}^{\mathrm{p}} \tag{30}$$

$$\dot{\bar{\mathbf{B}}} = \tfrac{2}{3}(1 - \beta) H \bar{\mathbf{D}}^{\mathrm{p}} \tag{31}$$

where H is the hardening modulus and β is the hardening ratio; both are considered to be functions of the equivalent plastic strain,

$$H = H(e^{\mathrm{p}}) \tag{32}$$

$$\beta = \beta(e^{\mathrm{p}}) \tag{33}$$

For values $\beta \in [0, 1]$ of the hardening ratio, equations (30)–(31) specify a combined isotropic–kinematic hardening rule, with $\beta = 1$ corresponding to purely isotropic hardening and $\beta = 0$ corresponding to purely kinematic hardening. The hardening modulus $H(e^{\mathrm{p}})$ is obtained as the instantaneous ratio of the Cauchy stress increment to the logarithmic strain increment in a uniaxial stress experiment, while the hardening ratio $\beta(e^{\mathrm{p}})$ is obtained by measuring the change in yield stress achieved in loading–unloading and then reloading uniaxial stress experiments.[34]

COMPUTATIONAL PROCEDURE

In a displacement-based finite element procedure for non-linear problems, the solution of the discretized equilibrium equations is obtained for each time step by an iterative technique. The result of each iteration is an estimate of the incremental displacements that are used to compute the stresses and other field variables at the integration points. If the nodal point forces corresponding to these stresses do not satisfy equilibrium to within a given tolerance, then the estimate of the incremental displacements is revised and the process repeated until convergence is achieved.[3]

We assume therefore that we are given (corresponding to a generic integration point) the deformation gradient ${}^{t}_{0}\mathbf{X}$ and the list of variables

$$\{{}^{t}\bar{\boldsymbol{\tau}},\ {}^{t}_{0}\mathbf{X}^{\mathrm{p}},\ {}^{t}e^{\mathrm{p}},\ {}^{t}\sigma,\ {}^{t}\bar{\mathbf{B}}\} \tag{34}$$

at time $\tau = t$, and the deformation gradient ${}^{t+\Delta t}_{\ \ 0}\mathbf{X}$ at time $\tau = t + \Delta t$.*

Our purpose is to develop a time integration algorithm to obtain

$$\{{}^{t+\Delta t}\bar{\boldsymbol{\tau}},\ {}^{t+\Delta t}_{\ \ 0}\mathbf{X}^{\mathrm{p}},\ {}^{t+\Delta t}e^{\mathrm{p}},\ {}^{t+\Delta t}\sigma,\ {}^{t+\Delta t}\bar{\mathbf{B}}\} \tag{35}$$

and the Cauchy stress ${}^{t+\Delta t}\boldsymbol{\tau}$ at time $t + \Delta t$.

The integration procedure for the present rate-independent model corresponds to the well-known 'radial return' algorithm of Wilkins[35] (see also Krieg and Key,[36] Krieg and Krieg[37] and Schreyer *et al.*[38]), and uses the effective-stress-function solution of Kojić and Bathe.[4,5]

* We are using here the notation of the book by Bathe[3]

Consider the evolution equation for the plastic deformation gradient

$$\dot{\mathbf{X}}^{\mathrm{p}} = \bar{\mathbf{D}}^{\mathrm{p}}\mathbf{X}^{\mathrm{p}} \tag{36}$$

We select the one-step implicit integration operator corresponding to the Euler backward method

$${}^{t+\Delta t}_{\ 0}\mathbf{X}^{\mathrm{p}} = \exp\,(\Delta t\,{}^{t+\Delta t}\bar{\mathbf{D}}^{\mathrm{p}})\,{}^{t}_{0}\mathbf{X}^{\mathrm{p}} \tag{37}$$

We note that this operator satisfies the consistency conditions

$$\lim_{\Delta t\to 0}\ {}^{t+\Delta t}_{\ 0}\mathbf{X}^{\mathrm{p}} = {}^{t}_{0}\mathbf{X}^{\mathrm{p}} \tag{38}$$

$$\lim_{\Delta t\to 0}\frac{\mathrm{d}}{\mathrm{d}\Delta t}\,{}^{t+\Delta t}_{\ 0}\mathbf{X}^{\mathrm{p}} = {}^{t}_{0}\dot{\mathbf{X}}^{\mathrm{p}} = {}^{t}\bar{\mathbf{D}}^{\mathrm{p}}\,{}^{t}_{0}\mathbf{X}^{\mathrm{p}} \tag{39}$$

and is first-order-accurate in Δt. Note that we could also use in equation (37) the α-method to here obtain a second-order-accurate time integration scheme (trapezoidal rule, $\alpha = 1/2$) useful in creep analysis.[3,5] However, in plasticity the Euler backward method is usually employed.

Taking the inverse in equation (37),

$$({}^{t+\Delta t}_{\ 0}\mathbf{X}^{\mathrm{p}})^{-1} = ({}^{t}_{0}\mathbf{X}^{\mathrm{p}})^{-1}\exp(-\Delta t\,{}^{t+\Delta t}\bar{\mathbf{D}}^{\mathrm{p}}) \tag{40}$$

and premultiplying by ${}^{t+\Delta t}_{\ 0}\mathbf{X}$ we obtain

$${}^{t+\Delta t}_{\ 0}\mathbf{X}^{\mathrm{e}} = \mathbf{X}^{\mathrm{e}}_{*}\exp(-\Delta t\,{}^{t+\Delta t}\bar{\mathbf{D}}^{\mathrm{p}}) \tag{41}$$

or equivalently,

$$\mathbf{X}^{\mathrm{e}}_{*} = {}^{t+\Delta t}_{\ 0}\mathbf{X}^{\mathrm{e}}\exp(\Delta t\,{}^{t+\Delta t}\bar{\mathbf{D}}^{\mathrm{p}}) \tag{42}$$

where

$$\mathbf{X}^{\mathrm{e}}_{*} = {}^{t+\Delta t}_{\ 0}\mathbf{X}({}^{t}_{0}\mathbf{X}^{\mathrm{p}})^{-1} \tag{43}$$

The quantity $\mathbf{X}^{\mathrm{e}}_{*}$ is the 'trial' elastic deformation gradient.

Premultiplying each side of equation (42) by its transpose we obtain

$$\mathbf{C}^{\mathrm{e}}_{*} = \exp(\Delta t\,{}^{t+\Delta t}\bar{\mathbf{D}}^{\mathrm{p}})\,{}^{t+\Delta t}\mathbf{C}^{\mathrm{e}}\exp(\Delta t\,{}^{t+\Delta t}\bar{\mathbf{D}}^{\mathrm{p}}) \tag{44}$$

where $\mathbf{C}^{\mathrm{e}}_{*} = (\mathbf{X}^{\mathrm{e}}_{*})^{\mathrm{T}}\mathbf{X}^{\mathrm{e}}_{*}$ and ${}^{t+\Delta t}\mathbf{C}^{\mathrm{e}} = ({}^{t+\Delta t}_{\ 0}\mathbf{X}^{\mathrm{e}})^{\mathrm{T}}\,{}^{t+\Delta t}_{\ 0}\mathbf{X}^{\mathrm{e}}$. By observing that

$$\mathbf{C}^{\mathrm{e}}_{*} = \exp(2\mathbf{E}^{\mathrm{e}}_{*}) \tag{45}$$

$${}^{t+\Delta t}\mathbf{C}^{\mathrm{e}} = \exp(2\,{}^{t+\Delta t}\mathbf{E}^{\mathrm{e}}) \tag{46}$$

equation (44) can be rewritten as

$$\exp(2\mathbf{E}^{\mathrm{e}}_{*}) = \exp(\Delta t\,{}^{t+\Delta t}\bar{\mathbf{D}}^{\mathrm{p}})\exp(2\,{}^{t+\Delta t}\mathbf{E}^{\mathrm{e}})\exp(\Delta t\,{}^{t+\Delta t}\bar{\mathbf{D}}^{\mathrm{p}}) \tag{47}$$

Taking the determinant on both sides of this equation (and recalling the identity $\det[\exp\mathbf{A}] = \exp[\mathrm{tr}\,\mathbf{A}]$), we obtain

$$\mathrm{tr}\,\mathbf{E}^{\mathrm{e}}_{*} = \mathrm{tr}\,{}^{t+\Delta t}\mathbf{E}^{\mathrm{e}} = \ln\,{}^{t+\Delta t}_{\ 0}J \tag{48}$$

where use was made of the fact that the flow rule (28) implies $\mathrm{tr}\,{}^{t+\Delta t}\bar{\mathbf{D}}^{\mathrm{p}} = 0$.

In view of (48) equation (47) can be written as

$$\exp(2\mathbf{E}^{\mathrm{e}\prime}_{*}) = \exp(\Delta t\,{}^{t+\Delta t}\bar{\mathbf{D}}^{\mathrm{p}})\exp(2\,{}^{t+\Delta t}\mathbf{E}^{\mathrm{e}\prime})\exp(\Delta t\,{}^{t+\Delta t}\bar{\mathbf{D}}^{\mathrm{p}}) \tag{49}$$

where $\mathbf{A}' = \mathrm{dev}\,\mathbf{A}$ indicates the deviatoric part.

The following result is now essential for the development of the algorithm.

Proposition 1. Let $\mathbf{E}$ and $\mathbf{D}$ be symmetric tensors and let ε be a scalar. Let $\mathbf{E}_*$ be the tensor defined by

$$\exp(2\mathbf{E}_*) = \exp(\varepsilon\mathbf{D}) \exp(2\mathbf{E}) \exp(\varepsilon\mathbf{D}) \tag{50}$$

then, if $\| \mathbf{E} \|$ and $\| \varepsilon\mathbf{D} \|$ are sufficiently small,

$$\mathbf{E}_* = \mathbf{E} + \varepsilon \left\{ \sum_{n=0}^{\infty} \frac{(-2)^n}{(n+1)} \sum_{r=0}^{n} \mathbf{E}_0^r (\mathbf{D} + \mathbf{D}\mathbf{E}_0 + \mathbf{E}_0\mathbf{D}) \mathbf{E}_0^{n-r} \right\} + \mathcal{O}(\| \varepsilon\mathbf{D} \|^2) \tag{51}$$

where $\mathbf{E}_0$ is given by

$$\mathbf{E}_0 = \tfrac{1}{2}(\exp(2\mathbf{E}) - \mathbf{1}) \tag{52}$$

and $\mathcal{O}(\| \varepsilon\mathbf{D} \|^2)$ is a tensor valued function of $\varepsilon\mathbf{D}$ such that

$$\lim_{\| \varepsilon\mathbf{D} \| \to 0} \frac{\mathcal{O}(\| \varepsilon\mathbf{D} \|^2)}{\| \varepsilon\mathbf{D} \|} = \mathbf{0} \tag{53}$$

The proof is given in the Appendix. ■

In view of (51), it follows that, to first-order in Δt, equation (49) is equivalent to

$$\mathbf{E}_*^{e'} = {}^{t+\Delta t}\mathbf{E}^{e'} + \Delta t\{{}^{t+\Delta t}\bar{\mathbf{D}}^p - \tfrac{1}{3}[[\mathbf{E}_0, {}^{t+\Delta t}\bar{\mathbf{D}}^p], \mathbf{E}_0] + \dots\} \tag{54}$$

where $\mathbf{E}_0$ is given by

$$\mathbf{E}_0 = \tfrac{1}{2}(\exp(2\,{}^{t+\Delta t}\mathbf{E}^{e'}) - \mathbf{1}) \tag{55}$$

It follows that for moderately small elastic strains equation (54) can be approximated by

$$\mathbf{E}_*^{e'} = {}^{t+\Delta t}\mathbf{E}^{e'} + \Delta t\, {}^{t+\Delta t}\bar{\mathbf{D}}^p \tag{56}$$

We note that for isotropic hardening plasticity or for combined isotropic–kinematic hardening cases where the stress and back stress tensors commute, equation (56) is exactly equivalent to (49), while otherwise the error is second-order in the elastic strains (when compared to unity).

It follows from equation (41) that the rotation tensors ${}^{t+\Delta t}_{\ \ 0}\mathbf{R}^e$ and $\mathbf{R}_*^e$ are related by

$${}^{t+\Delta t}_{\ \ 0}\mathbf{R}^e = \mathbf{R}_*^e \exp(\mathbf{E}_*^e) \exp(-\Delta t\, {}^{t+\Delta t}\bar{\mathbf{D}}^p) \exp(-{}^{t+\Delta t}\mathbf{E}^e) \tag{57}$$

It can be shown that to the same order of approximation implied by (56) we have

$${}^{t+\Delta t}_{\ \ 0}\mathbf{R}^e = \mathbf{R}_*^e \tag{58}$$

Let $\bar{\boldsymbol{\tau}}_*$ denote the 'trial rotated stress tensor', given by

$$\bar{\boldsymbol{\tau}}_* = \bar{\boldsymbol{\tau}}'_* + \tfrac{1}{3}(\operatorname{tr}\bar{\boldsymbol{\tau}}_*)\mathbf{1} \tag{59}$$

$$\operatorname{tr}\bar{\boldsymbol{\tau}}_* = 3\kappa \operatorname{tr}\mathbf{E}_*^e \tag{60}$$

$$\bar{\boldsymbol{\tau}}'_* = 2\mu\mathbf{E}_*^{e'} \tag{61}$$

We define by $\bar{\mathbf{S}}_* = \bar{\boldsymbol{\tau}}'_* - {}^t\bar{\mathbf{B}}$ and $s_* = \sqrt{\tfrac{3}{2}\bar{\mathbf{S}}_* \cdot \bar{\mathbf{S}}_*}$, respectively, the 'trial rotated effective stress' and the 'trial equivalent tensile stress'.

If $s_* < {}^t\sigma$, then the process is elastic, ${}^{t+\Delta t}\bar{\mathbf{D}}^p = \mathbf{0}$ and ${}^{t+\Delta t}\bar{\boldsymbol{\tau}} = \bar{\boldsymbol{\tau}}_*$ and hence the updated state is equal to the trial elastic state. When $s_* \geq {}^t\sigma$, the process is considered elasto-plastic. We assume in the following discussion that the process is elasto-plastic.

Using the stress–strain law (14) we then obtain from (56)

$$\bar{\boldsymbol{\tau}}'_* = {}^{t+\Delta t}\bar{\boldsymbol{\tau}}' + 2\mu\Delta t\,{}^{t+\Delta t}\bar{\mathbf{D}}^{\mathrm{p}} \tag{62}$$

The flow rule (28) and the evolution equation for the back stress (31) are integrated by the Euler backward operators

$$\Delta t\,{}^{t+\Delta t}\bar{\mathbf{D}}^{\mathrm{p}} = \sqrt{\tfrac{3}{2}}({}^{t+\Delta t}e^{\mathrm{p}} - {}^{t}e^{\mathrm{p}})\,{}^{t+\Delta t}\bar{\mathbf{N}} \tag{63}$$

$${}^{t+\Delta t}\bar{\mathbf{B}} = {}^{t}\bar{\mathbf{B}} + \tfrac{2}{3}[1 - {}^{t+\Delta t}\beta]\,{}^{t+\Delta t}H\Delta t\,{}^{t+\Delta t}\bar{\mathbf{D}}^{\mathrm{p}} \tag{64}$$

Combining equations (64) and (62) gives

$$\bar{\mathbf{S}}_* = {}^{t+\Delta t}\bar{\mathbf{S}} + \tfrac{2}{3}\{3\mu + [1 - {}^{t+\Delta t}\beta]\,{}^{t+\Delta t}H\}\Delta t\,{}^{t+\Delta t}\bar{\mathbf{D}}^{\mathrm{p}} \tag{65}$$

Substituting for $\Delta t\,{}^{t+\Delta t}\bar{\mathbf{D}}^{\mathrm{p}}$ from (63) we obtain

$$\bar{\mathbf{S}}_* = {}^{t+\Delta t}\bar{\mathbf{S}} + \sqrt{\tfrac{2}{3}}\{3\mu + [1 - {}^{t+\Delta t}\beta]\,{}^{t+\Delta t}H\}({}^{t+\Delta t}e^{\mathrm{p}} - {}^{t}e^{\mathrm{p}})\,{}^{t+\Delta t}\bar{\mathbf{N}} \tag{66}$$

which, in view of definition (23) leads to

$$\sqrt{\tfrac{3}{2}}\,\bar{\mathbf{S}}_* = [{}^{t+\Delta t}s + \{3\mu + [1 - {}^{t+\Delta t}\beta]\,{}^{t+\Delta t}H\}({}^{t+\Delta t}e^{\mathrm{p}} - {}^{t}e^{\mathrm{p}})]\,{}^{t+\Delta t}\bar{\mathbf{N}} \tag{67}$$

Taking the norm on both sides of this equation we obtain

$${}^{t+\Delta t}s + \{3\mu + [1 - {}^{t+\Delta t}\beta]\,{}^{t+\Delta t}H\}({}^{t+\Delta t}e^{\mathrm{p}} - {}^{t}e^{\mathrm{p}}) = s_* \tag{68}$$

The consistency condition (27) implies that when the process is elasto-plastic (when $\dot{e}^{\mathrm{p}} > 0$) $\dot{\Phi} = 0$, or equivalently, $\dot{\sigma} = \dot{s}$. It follows from the hardening rule for the deformation resistance (30) that

$$\dot{s} = \beta H \dot{e}^{\mathrm{p}} \tag{69}$$

This equation is integrated by the Euler backward operator

$${}^{t+\Delta t}s = {}^{t}s + ({}^{t+\Delta t}e^{\mathrm{p}} - {}^{t}e^{\mathrm{p}})\,{}^{t+\Delta t}\beta\,{}^{t+\Delta t}H \tag{70}$$

Combining equations (68) and (70) we obtain a single scalar equation in the equivalent plastic strain ${}^{t+\Delta t}e^{\mathrm{p}}$,

$$f({}^{t+\Delta t}e^{\mathrm{p}}) = [3\mu + H({}^{t+\Delta t}e^{\mathrm{p}})]({}^{t+\Delta t}e^{\mathrm{p}} - {}^{t}e^{\mathrm{p}}) - (s_* - {}^{t}s) = 0 \tag{71}$$

where $f({}^{t+\Delta t}e^{\mathrm{p}})$ is the 'effective-stress-function', here written in terms of the unknown effective plastic strain.[5]

For the particular case of a constant hardening modulus H,

$${}^{t+\Delta t}e^{\mathrm{p}} = {}^{t}e^{\mathrm{p}} + \frac{s_* - {}^{t}s}{3\mu + H} \tag{72}$$

Once the equivalent plastic strain ${}^{t+\Delta t}e^{\mathrm{p}}$ has been obtained, the equivalent tensile effective stress is evaluated using (70):

$${}^{t+\Delta t}s = {}^{t}s + ({}^{t+\Delta t}e^{\mathrm{p}} - {}^{t}e^{\mathrm{p}})\,{}^{t+\Delta t}\beta\,{}^{t+\Delta t}H \tag{73}$$

Note that while in an elasto-plastic process ${}^{t+\Delta t}e^{\mathrm{p}} - {}^{t}e^{\mathrm{p}}$ is always positive, ${}^{t+\Delta t}s - {}^{t}s$ is greater than or equal to zero. The latter case arises in the purely kinematic hardening case for which ${}^{t+\Delta t}\beta = 0$. Equations (71)–(73) are also appropriate for this limiting case.

With the values ${}^{t+\Delta t}e^{\mathrm{p}}$, ${}^{t+\Delta t}s$ and the unit normal

$${}^{t+\Delta t}\bar{\mathbf{N}} = \sqrt{\frac{3}{2}}\frac{\bar{\mathbf{S}}_*}{s_*} \tag{74}$$

obtained from equation (67), the elasto-plastic state can be updated by using equations (63), (62), (64), (48) and (37).

The complete computational procedure is summarized in the following equations.

1. The trial elastic state.

(a) Obtain the trial elastic deformation gradient

$$\mathbf{X}_*^e = {}^{t+\Delta t}_0\mathbf{X}({}^t_0\mathbf{X}^p)^{-1} \tag{75}$$

(b) Perform the polar decomposition

$$\mathbf{X}_*^e = \mathbf{R}_*^e \mathbf{U}_*^e \tag{76}$$

(c) Obtain the trial elastic strain tensor

$$\mathbf{E}_*^e = \ln \mathbf{U}_*^e \tag{77}$$

(d) Obtain the trial elastic stress tensor and the trial elastic effective stress tensor

$$\operatorname{tr} \bar{\boldsymbol{\tau}}_* = 3\kappa \operatorname{tr} \mathbf{E}_*^e \tag{78}$$

$$\bar{\boldsymbol{\tau}}'_* = 2\mu \mathbf{E}_*^{e\prime} \tag{79}$$

$$\bar{\mathbf{S}}_* = \bar{\boldsymbol{\tau}}'_* - {}^t\bar{\mathbf{B}} \tag{80}$$

(e) Obtain the trial equivalent tensile stress

$$s_* = \sqrt{\tfrac{3}{2}\bar{\mathbf{S}}_* \cdot \bar{\mathbf{S}}_*} \tag{81}$$

2. Check for elastic process. Effective stress function.

(a) If $s_* < {}^t\sigma$ then the process is elastic and

$${}^{t+\Delta t}s = s_* \tag{82}$$

$${}^{t+\Delta t}\sigma = {}^t\sigma \tag{83}$$

$${}^{t+\Delta t}\mathbf{E}^e = \mathbf{E}_*^e \tag{84}$$

$${}^{t+\Delta t}\bar{\boldsymbol{\tau}} = \bar{\boldsymbol{\tau}}_* \tag{85}$$

$${}^{t+\Delta t}\bar{\mathbf{B}} = {}^t\bar{\mathbf{B}} \tag{86}$$

$${}^{t+\Delta t}\boldsymbol{\tau} = \mathbf{R}_*^e \bar{\boldsymbol{\tau}}_* (\mathbf{R}_*^e)^{\mathrm{T}} \tag{87}$$

$$\text{EXIT} \tag{88}$$

Else, the process is elasto-plastic. Obtain ${}^{t+\Delta t}e^p$ by solving

$$f({}^{t+\Delta t}e^p) = [3\mu + H({}^{t+\Delta t}e^p)]({}^{t+\Delta t}e^p - {}^te^p) - (s_* - {}^ts) = 0 \tag{89}$$

3. Return mapping.

(a) Update the equivalent tensile effective stress and the deformation resistance

$${}^{t+\Delta t}s = {}^ts + ({}^{t+\Delta t}e^p - {}^te^p)\,{}^{t+\Delta t}\beta\,{}^{t+\Delta t}H \tag{90}$$

$${}^{t+\Delta t}\sigma = {}^{t+\Delta t}s \tag{91}$$

(b) Update the unit normal

$${}^{t+\Delta t}\bar{\mathbf{N}} = \sqrt{\frac{3}{2}}\frac{\bar{\mathbf{S}}_*}{s_*} \tag{92}$$

(c) Update the modified plastic stretching tensor

$$^{t+\Delta t}\bar{\mathbf{D}}^{\mathrm{p}} = \sqrt{\frac{3}{2}}\frac{(^{t+\Delta t}e^{\mathrm{p}} - {}^{t}e^{\mathrm{p}})}{\Delta t}\,{}^{t+\Delta t}\bar{\mathbf{N}} \tag{93}$$

(d) Update the rotated stress deviator and the back stress

$$^{t+\Delta t}\bar{\boldsymbol{\tau}}' = \bar{\boldsymbol{\tau}}'_{*} - 2\mu\Delta t\,{}^{t+\Delta t}\bar{\mathbf{D}}^{\mathrm{p}} \tag{94}$$

$$^{t+\Delta t}\bar{\mathbf{B}} = {}^{t}\bar{\mathbf{B}} + \tfrac{2}{3}[1 - {}^{t+\Delta t}\beta]\,{}^{t+\Delta t}H\,\Delta t\,{}^{t+\Delta t}\bar{\mathbf{D}}^{\mathrm{p}} \tag{95}$$

(e) Update the rotated stress tensor

$$^{t+\Delta t}\bar{\boldsymbol{\tau}} = {}^{t+\Delta t}\bar{\boldsymbol{\tau}}' + \tfrac{1}{3}(\mathrm{tr}\,\bar{\boldsymbol{\tau}}_{*})\mathbf{1} \tag{96}$$

(f) Update the Cauchy stress

$$^{t+\Delta t}\boldsymbol{\tau} = \mathbf{R}^{\mathrm{e}}_{*}\,{}^{t+\Delta t}\bar{\boldsymbol{\tau}}(\mathbf{R}^{\mathrm{e}}_{*})^{\mathrm{T}} \tag{97}$$

(g) Update the plastic deformation gradient

$$^{t+\Delta t}_{\ 0}\mathbf{X}^{\mathrm{p}} = \exp(\Delta t\,{}^{t+\Delta t}\bar{\mathbf{D}}^{\mathrm{p}})\,{}^{t}_{0}\mathbf{X}^{\mathrm{p}} \tag{98}$$

We refer to References 3, 5 and 9 for the details of the actual computations. Note, for example, that we use instead of the deformation gradient the inverse of this gradient, and that the solution scheme is, of course, applicable to fully 3-D analysis and 2-D plane strain and axisymmetric solutions, and can also directly be used for 2-D plane stress and shell solutions with the effective-stress-function procedure discussed in Reference 39.

SOME SIMPLE RESPONSE PREDICTIONS

The objective of this section is to show some simple response predictions using the constitutive model presented in this paper. We also compare these results with the response predicted using a similar elasto-plastic model based on the additive decomposition of the strain tensor.[9]

Two basic deformation patterns are considered:

1. An in-plane isochoric deformation with no rotation of the principal stretch directions, given by the deformation gradient

$$[\mathbf{X}] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1/\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{99}$$

where $\lambda \in [1, 1{\cdot}5]$.

2. A simple shearing deformation given by the deformation gradient

$$[\mathbf{X}] = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{100}$$

where $\gamma \in [0, 0{\cdot}5]$.

In these solutions we consider strains to about 50 per cent. Table I summarizes the material properties selected for the examples. Figures 1–5 show resulting stress components.

As exemplified in Figure 1, in the case of the deformation pattern 1 the model based on the product decomposition and the model based on the additive decomposition predict the same stresses. In fact it can be proved that for deformation histories for which the total and elastic strain tensors commute and no plastic rotations are involved both kinematic descriptions are equivalent.

Figures 2–5 show the results of the simple shearing case. Within the range of applicability of a combined isotropic–kinematic hardening model (our objective is to model strains to about 40 per

Table I. Material constants

Shear modulus μ	76·92 MPa
Bulk modulus κ	166·67 MPa
Flow stress σ_0	0·75 MPa
Hardening modulus H	2·00 MPa

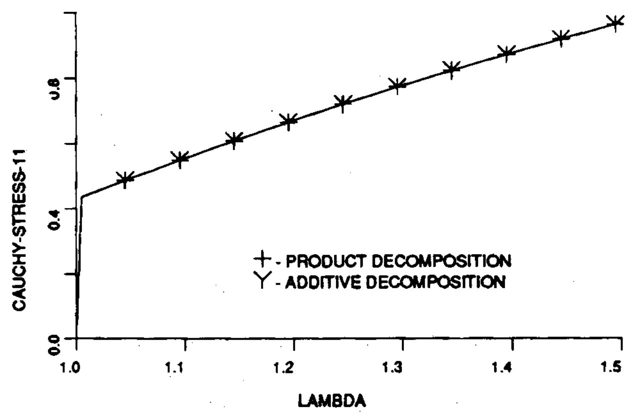

Figure 1. Cauchy stress component τ_{11} for deformation pattern 1 and any hardening ratio β

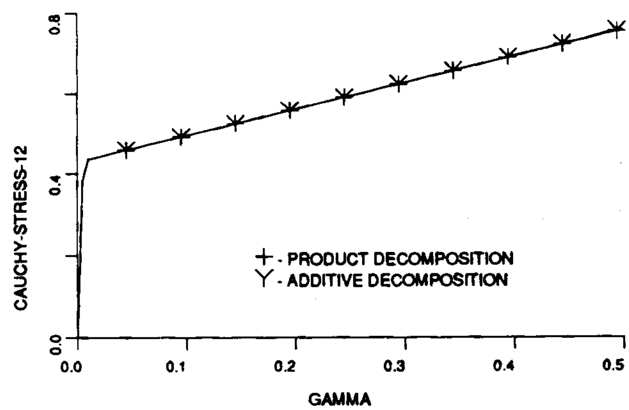

Figure 2. Cauchy stress component τ_{12} for deformation pattern 2 and isotropic hardening ($\beta = 1$)

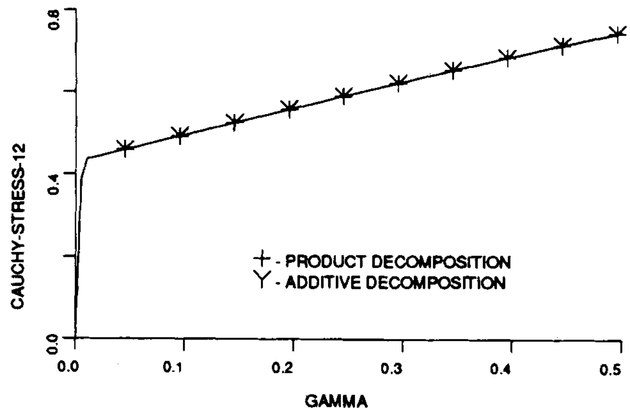

Figure 3. Cauchy stress component τ_{12} for deformation pattern 2 and kinematic hardening ($\beta = 0$)

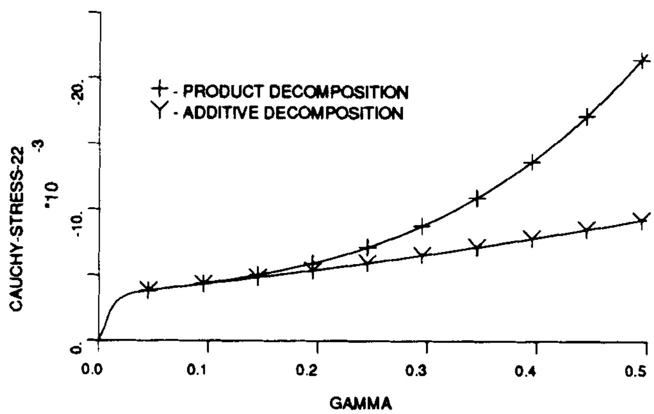

Figure 4. Cauchy stress component τ_{22} for deformation pattern 2 and isotropic hardening ($\beta = 1$)

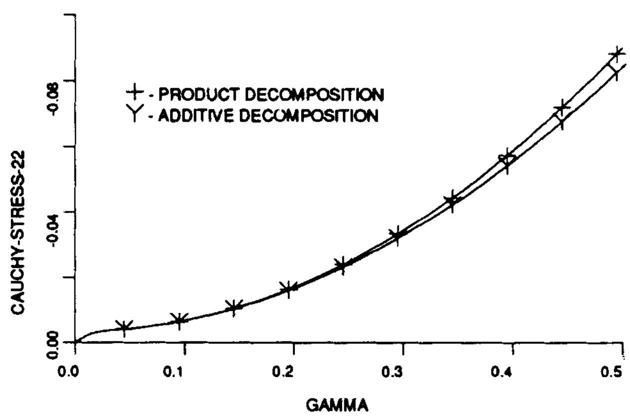

Figure 5. Cauchy stress component τ_{22} for deformation pattern 2 and kinematic hardening ($\beta = 0$)

cent), the shearing (dominant) stress is appreciably the same. Some difference occurs in the normal stress, with the model based on the product decomposition predicting larger values.

CONCLUDING REMARKS

The objective in this paper was to present a hyperelastic-based large strain formulation using the product decomposition of the deformation gradient for finite element elasto-plastic analysis. The formulation uses the Hencky strain and is applicable to combined isotropic and kinematic hardening.

An important feature of the formulation is that it is a total formulation. Hence it is inherently objective, large integration steps can be used and it does not lead to non-conservative response predictions in elastic solutions (as do the hypoelastic-based models).

The formulation gives physically realistic response predictions and, since it contains the basic ingredients of a numerically efficient solution procedure, is very attractive for large strain elasto-plastic finite element analysis.

APPENDIX

Proof of Proposition 1

For any second-order tensor $\mathbf{X}$ and any second-order tensor $\mathbf{Y}$ such that $\|\mathbf{Y}\| < 1$, the tensor exponential $\exp\mathbf{X}$ and the tensor logarithm $\ln(\mathbf{1}+\mathbf{Y})$ are defined by the series (see for example Reference 40)

$$\exp\mathbf{X} = \sum_{n=0}^{\infty} \frac{\mathbf{X}^n}{n!} \tag{101}$$

$$\ln(\mathbf{1}+\mathbf{Y}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)} \mathbf{Y}^{n+1} \tag{102}$$

It follows from (101) that

$$\exp(\varepsilon\mathbf{D}) = \mathbf{1} + \varepsilon\mathbf{D} + \mathcal{O}(\|\varepsilon\mathbf{D}\|^2) \tag{103}$$

where $\mathcal{O}(\|\varepsilon\mathbf{D}\|^2)$ is a tensor-valued function of the scalar ε such that

$$\lim_{\varepsilon\to 0} \frac{\mathcal{O}(\|\varepsilon\mathbf{D}\|^2)}{\varepsilon} = \mathbf{0} \tag{104}$$

Then, in view of (103), equation (50) can be written as

$$\exp(2\mathbf{E}_*) = [\mathbf{1} + \varepsilon\mathbf{D} + \mathcal{O}(\|\varepsilon\mathbf{D}\|^2)]\exp(2\mathbf{E})[\mathbf{1} + \varepsilon\mathbf{D} + \mathcal{O}(\|\varepsilon\mathbf{D}\|^2)] \tag{105}$$

or equivalently

$$\exp(2\mathbf{E}_*) = \exp(2\mathbf{E}) + \varepsilon(\mathbf{D}\exp(2\mathbf{E}) + \exp(2\mathbf{E})\mathbf{D}) + \mathcal{O}(\|\varepsilon\mathbf{D}\|^2) \tag{106}$$

and substituting for $\exp(2\mathbf{E})$ in terms of (52),

$$\exp(2\mathbf{E}_*) = \mathbf{1} + 2\mathbf{E}_0 + 2\varepsilon(\mathbf{D} + \mathbf{D}\mathbf{E}_0 + \mathbf{E}_0\mathbf{D}) + \mathcal{O}(\|\varepsilon\mathbf{D}\|^2) \tag{107}$$

We note next that if $\|\mathbf{E}_0\|$ and $\|\varepsilon\mathbf{D}\|$ are sufficiently small then the series expansion (102) is applicable, and we have

$$2\mathbf{E}_* = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)} [2\mathbf{E}_0 + 2\varepsilon(\mathbf{D} + \mathbf{D}\mathbf{E}_0 + \mathbf{E}_0\mathbf{D}) + \mathcal{O}(\|\varepsilon\mathbf{D}\|^2)]^{n+1} \tag{108}$$

Considering that

$$[2\mathbf{E}_0 + 2\varepsilon(\mathbf{D} + \mathbf{D}\mathbf{E}_0 + \mathbf{E}_0\mathbf{D}) + \mathcal{O}(\|\varepsilon\mathbf{D}\|^2)]^{n+1}$$

$$= (2\mathbf{E}_0)^{n+1} + 2^{n+1}\varepsilon \sum_{r=0}^{n} \mathbf{E}_0^r(\mathbf{D} + \mathbf{D}\mathbf{E}_0 + \mathbf{E}_0\mathbf{D})\mathbf{E}_0^{n-r} + \mathcal{O}(\|\varepsilon\mathbf{D}\|^2) \tag{109}$$

equation (108) can be written as

$$2\mathbf{E}_* = \ln(\mathbf{1} + 2\mathbf{E}_0) + 2\varepsilon \sum_{n=0}^{\infty} \frac{(-2)^n}{(n+1)} \sum_{r=0}^{n} \mathbf{E}_0^r(\mathbf{D} + \mathbf{D}\mathbf{E}_0 + \mathbf{E}_0\mathbf{D})\mathbf{E}_0^{n-r} + \mathcal{O}(\|\varepsilon\mathbf{D}\|^2) \tag{110}$$

or equivalently, using (52),

$$\mathbf{E}_* = \mathbf{E} + \varepsilon\mathbf{A} + \mathcal{O}(\|\varepsilon\mathbf{D}\|^2) \tag{111}$$

where the tensor $\mathbf{A}$ is given by

$$\mathbf{A} = \sum_{n=0}^{\infty} \frac{(-2)^n}{(n+1)} \sum_{r=0}^{n} \mathbf{E}_0^r(\mathbf{D} + \mathbf{D}\mathbf{E}_0 + \mathbf{E}_0\mathbf{D})\mathbf{E}_0^{n-r} \tag{112}$$

REFERENCES

1. K. J. Bathe, E. Ramm and E. L. Wilson, 'Finite element formulations for large deformation dynamic analysis', *Int. j. numer. methods eng.*, **9**, 353–386 (1975).
2. K. J. Bathe and E. L. Wilson, 'NONSAP—A general finite element program for nonlinear dynamic analysis of complex structures', in *Proc. Second Conf. on Structural Mechanics in Reactor Technology*, Berlin, Sept. 1973.
3. K. J. Bathe, *Finite Element Procedures in Engineering Analysis*. Prentice-Hall, Englewood Cliffs, N.J., 1982.
4. K. J. Bathe, A. Chaudhary, E. Dvorkin and M. Kojić, 'On the solution of nonlinear finite element equations', in F. Damjanić *et al.* (ed.), *Proc. Int. Conf. on Computer-Aided Analysis and Design of Concrete Structures I*, Pineridge Press, Swansea, U.K., 1984, pp. 289–299.
5. M. Kojić and K. J. Bathe, 'The "effective-stress-function" algorithm for thermo-elasto-plasticity and creep', *Int. j. numer. methods eng.*, **24**, 1509–1532 (1987).
6. A. E. Green and P. M. Naghdi, 'A general theory of an elastic–plastic continuum', *Arch. Rat. Mech. Anal.*, **18**, 251–281 (1965).
7. P. Perzyna and W. Wojno, 'On the constitutive equations for elastic/viscoplastic materials at finite strain', *Archwm. Mech. Stosow.*, **18**, 85–100 (1966).
8. P. Perzyna and W. Wojno, 'Thermodynamics of a rate sensitive plastic material', *Archwm. Mech. Stosow.*, **20**, 499–510 (1968).
9. W. D. Rolph III and K. J. Bathe, 'On a large strain finite element formulation for elasto-plastic analysis', in K. J. Willam (ed.), *Constitutive Equations: Macro and Computational Aspects*, AMD, ASME, New York, 1984, pp. 131–147.
10. K. J. Bathe, M. Kojić and J. Walczak, 'Some developments in methods for large strain elasto-plastic analysis', in D. R. J. Owen *et al.* (eds.), *Computational Plasticity: Models, Software and Applications I*, Pineridge Press, Swansea, U.K., 1987, pp. 263–279.
11. A. L. Eterovic and K. J. Bathe, 'A note on the use of the additive decomposition of the strain tensor in finite deformation inelasticity', Report 89-2, Finite Element Research Group, Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, MA, 1989.
12. E. H. Lee and D. T. Liu, 'Finite strain elastic–plastic theory particularly for plane wave analysis', *J. Appl. Phys.*, **38**, 19–27 (1967).
13. E. H. Lee, 'Elastic–plastic deformation at finite strains', *J. Appl. Mech. ASME*, **36**, 1–6 (1969).
14. J. Mandel, 'Equations constitutives et directeurs dans les milieux plastiques et viscoplastiques', *Int. J. Solids Struct.*, **9**, 725–740 (1973).
15. J. Mandel, 'Thermodynamics and plasticity', in J. J. Delgado Domingos *et al.* (eds.), *Foundations of Continuum Thermodynamics*, Wiley, New York, 1973, pp. 283–304.
16. S. Nemat-Nasser, 'On finite deformations elasto-plasticity', *Int. J. Solids Struct.*, **18** (1982).
17. R. J. Asaro, 'Micromechanics of crystals and polycrystals', *Adv. Appl. Mech.*, **23**(1) (1983).
18. D. R. J. Owen, E. Hinton and E. Oñate (eds.), *Computational Plasticity: Models, Software and Applications I*, Pineridge Press, Swansea, U.K., 1987.
19. M. Kojić and K. J. Bathe, 'Studies of finite element procedures—Stress solution of a closed elastic strain path with stretching and shearing using the updated Lagrangian Jaumann formulation', *Comp. Struct.*, **26**, 175–179 (1987).

20. T. J. R. Hughes and J. Winget, 'Finite rotation effects in numerical integration of rate constitutive equations arising in large-deformation analysis', *Int. j. numer. methods eng.*, **15**, 1862–1867 (1980).
21. J. C. Nagtegaal and J. E. De Jong, 'Some computational aspects of elastic–plastic large strain analysis', *Int. j. numer. methods eng.*, **17**, 15–41 (1981).
22. J. C. Nagtegaal, 'On the implementation of inelastic constitutive equations with special reference to large deformation problems', *Comp. Methods Appl. Mech. Eng.*, **33**, 469–484 (1982).
23. R. Rubinstein and S. N. Atluri, 'Objectivity of incremental constitutive relations over finite time steps in computational finite deformation analyses', *Comp. Methods Appl. Mech. Eng.*, **36**, 277–290 (1983).
24. K. W. Reed and S. N. Atluri, 'Constitutive modelling and computational implementation for finite strain plasticity', *Int. J. Plasticity*, **1**, 63–87 (1985).
25. J. C. Simo, 'A framework for finite strain elastoplasticity based on maximum plastic dissipation and the multiplicative decomposition: Part I. Continuum formulation', *Comp. Methods Appl. Mech. Eng.*, **66**, 199–219 (1988).
26. J. C. Simo, 'A framework for finite strain elastoplasticity based on maximum plastic dissipation and the multiplicative decomposition: Part II. Computational aspects', *Comp. Methods Appl. Mech. Eng.*, **68**, 1–31 (1988).
27. G. Weber and L. Anand, 'Finite deformation constitutive equations, and a time integration procedure for isotropic, hyperelastic–viscoplastic solids', *Comp. Methods Appl. Mech. Eng.*, accepted for publication.
28. J. R. Rice, 'Continuum mechanics and thermodynamics of plasticity in relation to microscale deformation mechanisms', in A. S. Argon (ed.), *Constitutive Equations in Plasticity*, The M.I.T. Press, Cambridge, MA, 1975, pp. 23–79.
29. R. Hill, 'Aspects of invariance in solid mechanics', in C.-S. Yih (ed.), *Advances in Applied Mechanics, Vol. 18*, Academic Press, New York, 1978, pp. 1–75.
30. B. D. Coleman and W. Noll, 'The thermodynamics of elastic materials with heat conduction and viscosity', *Arch. Rat. Mech. Anal.*, **13**, 245–261 (1963).
31. J. Lubliner, 'A maximum-dissipation principle in generalized plasticity', *Acta Mech.*, **52**, 225–237 (1984).
32. J. Lubliner, 'Normality rules in large-deformation plasticity', *Mech. Mat.*, **5**, 29–34 (1986).
33. D. G. Luenberger, *Linear and Nonlinear Programming*, 2nd edn, Addison-Wesley, Reading, MA, 1984.
34. R. Hill, *The Mathematical Theory of Plasticity*, Oxford University Press, Oxford, 1983.
35. M. L. Wilkins, 'Calculation of elastic–plastic flow', in B. Alder *et al.* (eds.), *Methods of Computational Physics, 3*, Academic Press, New York, 1964.
36. R. D. Krieg and S. W. Key, 'Implementation of a time independent plasticity theory into structural computer programs', in J. A. Stricklin and K. J. Saczalski (eds.), *Constitutive Equations in Viscoplasticity: Computational and Engineering Aspects*, AMD-Vol. 20, ASME, New York, 1976, pp. 125–137.
37. R. D. Krieg and D. B. Krieg, 'Accuracies of numerical solution methods for elastic–perfectly plastic model', *J. Press. Vess. Technol. ASME*, **99**, 510–515 (1977).
38. H. L. Schreyer, R. F. Kulak and J. M. Kramer, 'Accurate numerical solutions for elastic–plastic models', *J. Press. Vess. Technol. ASME*, **101**, 226–234 (1979).
39. M. Kojić and K. J. Bathe, 'Thermo-elastic–plastic and creep analysis of shell structures', *Comp. Struct.*, **26**, 135–143 (1987).
40. M. W. Hirsch and S. Smale, *Differential Equations, Dynamical Systems and Linear Algebra*, Academic Press, New York, 1974.