Journal of Applied Mechanics

Brief Notes

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A Simple and Effective Pipe Elbow Element—Pressure Stiffening Effects

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1 Introduction

In two previous communications we presented the formulation of a simple and effective pipe elbow element, which can be employed to model the variation of ovalization along an elbow, and interaction effects between elbows of different curvatures and elbows and straight piping sections [1, 2]. The objective in this Brief Note is to show how the element formulation can also be extended in a very simple way to include internal pressure stiffening effects. As has been established experimentally and theoretically, the effect of the internal pressure on the stiffness of an elbow section can be significant when relatively thin pipes are considered. In the presentation that follows we assume that the reader is familiar with our two earlier papers, and we concentrate only on the additional evaluations necessary to include the pressure stiffening effects.

2 Additional Term in Variational Formulation

Consider the elbow in Fig. 1 subjected to an internal pressure p. As the cross section of the pressurized bend deforms due to external loading the work of the pressure acting against the change in the cross-sectional area must be considered. Thus, the potential of the pressure to be added to the variational indicator (equation (17) of reference [2]) is

$$W_{pr} = -\int_{-1}^{+1}\int_{0}^{2\pi} p\,\frac{(R-a\cos\phi)\theta}{2}\quad dA(r,\phi)\,dr$$

where p is the internal pressure, $(R - a\cos\phi)$ is the longitudinal arc length of the midsurface of the bend, r is the isoparametric longitudinal coordinate, and $dA(r, \phi)$ is the differential change in the cross-sectional area of the pipe bend. This area quantity can conveniently be evaluated by using an auxiliary coordinate system $\bar{x}$, $\bar{y}$ to measure the coordinates of the points A and B on the midsurface of the bend (see Fig. 1). Let $A - B$ be the differential segment $ad\phi$ of the midsurface prior to deformation and $A' - B'$ be the same segment after deformation, then

$$dA = \frac{1}{2}\left((\bar{x}_A - \bar{x}_{B'})(\bar{y}_{A'} - \bar{y}_B) - (\bar{y}_A - \bar{y}_{B'})(\bar{x}_{A'} - \bar{x}_B) \right) \quad (1)$$

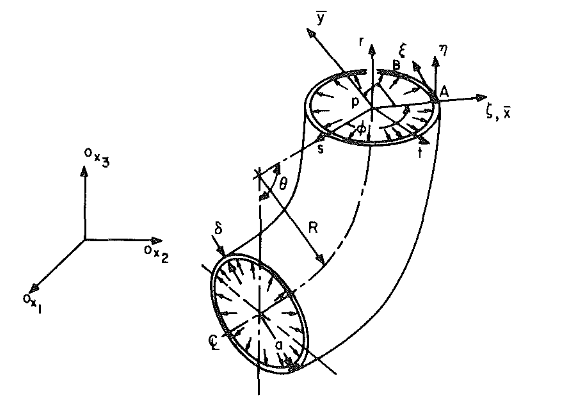

Fig. 1 Coordinate systems used in the elbow element formulation

where, with the displacements w_ξ and w_ζ into the ξ and ζ directions,

$$\begin{aligned}
&\bar{x}_A = a, \quad \bar{y}_A = 0\\
&\bar{x}_B = a\cos d\phi, \quad \bar{y}_B = a\sin d\phi\\
&\bar{x}_{A'} = a + w_\zeta, \quad \bar{y}_{A'} = w_\xi \qquad (2)\\
&\bar{x}_{B'} = (a + w_\zeta + dw_\zeta)\cos d\phi - (w_\xi + dw_\xi)\sin d\phi\\
&\bar{y}_{B'} = (a + w_\zeta + dw_\zeta)\sin d\phi + (w_\xi + dw_\xi)\cos d\phi
\end{aligned}$$

Substituting from equation (2) into equation (1) we obtain

$$dA = \frac{1}{2}\left[\underbrace{2\,a\,w_\zeta}_{\text{TERM1}} + w_\zeta\underbrace{\left(w_\zeta + \frac{dw_\xi}{d\phi}\right)}_{\text{TERM2}} + w_\xi^2 - \underbrace{w_\xi\frac{dw_\zeta}{d\phi}}_{\text{TERM3}}\right] d\phi \quad (3)$$

Considering equation (3) we now note that TERM1 will not contribute to W_{pr} because the integral of w_ζ over ϕ with the ovalization displacement assumption used (see equation (8) below) is zero. TERM2 is zero because we assume that the circumferential strains vanish at the midsurface of the bend (see equation (1) of reference [1]), so that to first order

$$w_\zeta = -\frac{dw_\xi}{d\phi} \quad (4)$$

To evaluate TERM3, which is of second order, we also use that the length of the arc $A - B$ shall be equal to the length of the arc $A' - B'$ but now include second-order terms. Hence, using equation (2) we assume

$$\begin{aligned}
(a\,d\phi)^2 = &\{dw_\zeta - (w_\xi + dw_\xi)d\phi\}^2\\
&+ \{(a + w_\zeta + dw_\zeta)d\phi + dw_\xi\}^2 \qquad (5)
\end{aligned}$$

and obtain with equation (4)

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PARAMETER	BEND 1	BEND 2	BEND 3	BEND 4
R	45.	14.25	12.4	10
δ	.5	.016	.020	.016
$\bar{a}$	15.	1.5	1.75	1.5
λ	.10483	.10623	.08489	.074545

(a) Bend geometric parameters (all dimensions in inches)

Boundary Conditions:

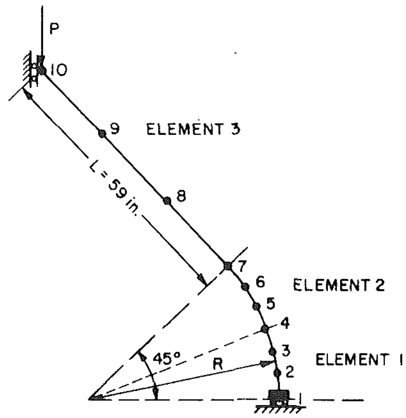
At node 1:

—no cross-sectional rotation, $\gamma_1 = 0$.

$-\dfrac{d w_\xi}{dx} = 0$

At nodes 4 and 7:

—ovalization and derivative of ovalization continuous



(b) Finite Element Model Used

Fig. 2 Pressurized bends considered in analyses. For definition of parameters see Figs. 1 and 3.

$$w_\xi \frac{dw_\xi}{d\phi} = \frac{1}{2}\left[w_\xi^2 + \left(\frac{d^2 w_\xi}{d\phi^2}\right)^2\right] \tag{6}$$

so that

$$W_{pr} = \frac{-p}{8}\int_{-1}^{+1}\int_0^{2\pi}\left(w_\xi^2 - \left(\frac{d^2 w_\xi}{d\phi^2}\right)^2\right)(R - a\cos\phi)\,\theta\, d\phi\, dr \tag{7}$$

3 Finite Element Matrix

The finite element stiffness matrix accounting for the internal pressure effect is obtained by substituting the ovalization displacement interpolations into the expression for W_{pr}, and including the term in the total potential Π when invoking the stationarity of Π. Using the ovalization displacement interpolations of our elbow element [1, 2]

$$w_\xi(r,\phi) = \sum_{m=1}^{N_c}\sum_{k=1}^{4} h_k\, c_m^k \sin 2m\phi + \sum_{m=1}^{N_d}\sum_{k=1}^{4} h_k\, d_m^k \cos 2m\phi \tag{8}$$

where N_c and N_d are the number of ovalization displacement components to be included, we obtain (similar to the calculation of the penalty matrices in reference [2]) the following pressure stiffness matrix,

$$\mathbf{K}_{pr} = \frac{p}{2}\int_{-1}^{+1}\int_0^{2\pi}[\mathbf{G}_{p1}^T\,\mathbf{G}_{p1} - \mathbf{G}_{p2}^T\,\mathbf{G}_{p2}]\left[\frac{(R - a\cos\phi)\,\theta}{2}\right] d\phi\, dr \tag{9}$$

where

$$\mathbf{G}_{p1} = [\ldots\ldots\ldots\bar{a}_1^k\;\bar{a}_2^k\;\bar{a}_3^k\;\bar{b}_1^k\;\bar{b}_2^k\;\bar{b}_3^k\ldots\ldots\ldots] \tag{10}$$

$$\bar{a}_m^k = -(2m)^2\, h_k \sin 2m\phi$$

$$\bar{b}_m^k = -(2m)^2\, h_k \cos 2m\phi$$

$$\mathbf{G}_{p2} = [\ldots\ldots\ldots\tilde{a}_1^k\;\tilde{a}_2^k\;\tilde{a}_3^k\;\tilde{b}_1^k\;\tilde{b}_2^k\;\tilde{b}_3^k\ldots\ldots\ldots] \tag{11}$$

$$\tilde{a}_m^k = h_k \sin 2m\phi$$

$$\tilde{b}_m^k = h_k \cos 2m\phi$$

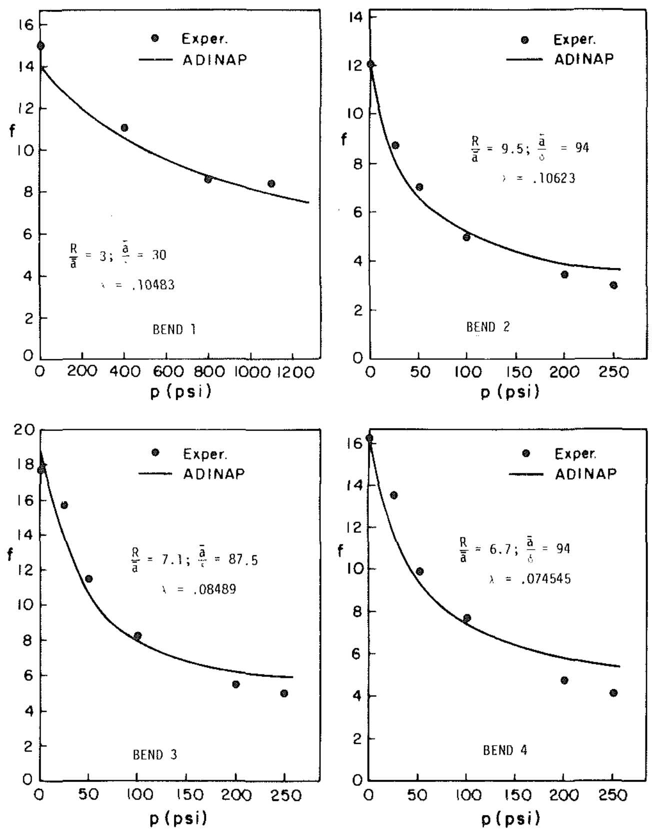


Fig. 3 Stiffening effects on the pipes of Fig. 2 due to internal pressure; $f \dfrac{4\gamma_7}{M}\dfrac{a^3 E\delta}{R}$; γ_7 = cross-sectional rotation at node 7; $\bar{a} = a + \delta/2$; $\lambda = R\delta/(a^2\sqrt{1-\nu^2})$; $\nu = 0.3$; E and ν are Young's modulus and Poisson's ratio.

and $\mathbf{K}_{pr}$ is defined corresponding to the ovalization degrees of freedom

$$\mathbf{U}^T = [\ldots\ldots\ldots c_1^k\; c_2^k\; c_3^k\; d_1^k\; d_2^k\; d_3^k\ldots\ldots\ldots] \tag{12}$$

Referring to equations (9)–(11) we note that the evaluation of $\mathbf{K}_{pr}$ is quite inexpensive because only relatively simple integrations need be performed and the terms in $\mathbf{G}_{p1}$ and $\mathbf{G}_{p2}$ are similar.

The complete equilibrium equations corresponding to an elbow element are then

$$(\mathbf{K}_L + \mathbf{K}_p + \mathbf{K}_{pr})\,\mathbf{U} = \mathbf{R} \tag{13}$$

where $\mathbf{K}_L$ is the stiffness matrix of the elbow without internal pressure and continuity effects, as defined in reference [1], $\mathbf{K}_p$ is the penalty matrix to account for continuity effects, as defined in reference [2], and $\mathbf{R}$ is the vector of external nodal point loads [1].

4 Sample Analyses

To indicate the applicability of the foregoing analysis procedure we present the results obtained in the analyses of the bends described in Fig. 2. As shown, in these analyses three elbow elements (2 curved and 1 straight) were used, and continuity conditions were imposed between the curved and the straight sections. Figure 3 shows the calculated flexibilities for the bends and experimental results reported in references [3, 4]. It is noted that the flexibilities computed in this study are quite close to the experimental values except for higher pressures. The discrepancy at larger pressures is probably due to neglecting in the elbow formulation the midsurface strains in the ξ – direction (assumption 3, Section 2.1 of reference [1]).

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Optimization of Reinforcement for a Class of Openings in Plate Structures

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A numerical/analytical procedure has been developed which yields the optimum amount of reinforcement for a given hole shape in a large elastic plate under prescribed boundary stresses. This procedure is based on determining the usual two complex potentials which describe the entire stress field, constructing the strain energy density function in terms of the unknown amount of reinforcement, integrating this function around the opening boundary, and finally minimizing this integral with respect to the reinforcement. The method is first developed for a general hole shape and then demonstrated in some detail for a circular and a square-like opening.

Introduction

The use of reinforcements at the boundaries of openings is standard practice in ship and aircraft construction. The amount of reinforcement to be used is determined, in most cases, by somewhat arbitrary decisions regarding the percentage reduction it produces in the boundary stress maximums. Although intrinsically there is nothing wrong with this approach, it seems to lack a rational basis or criterion for helping the structural designer select a specific amount of reinforcement. The procedure presented in this paper is an effort to provide such a basis. Rather than determining the stress field corresponding to a given geometry of the opening and of the reinforcement, this procedure seeks to determine the reinforcement that minimizes a certain meaningful integral related to the boundary strain energy. In this way it is an inverse elasticity problem. Some investigators [1-3], in studying the related problem of optimizing unreinforced notch shapes in plates, have concluded that uniform tangential stress at the notch boundary would, in general, lead to the smallest stress concentrations. Intuitively, it appears justified to assume that in the case of reinfored notch boundaries the requirement of uniformity of boundary stresses and/or stress related quantities, such as strain energy etc., at the notch boundary would lead to more desirable designs. The optimization rationale used in the present procedure is based on this argument.

In general, stress analysis of noncircular opening reinforced with a thin member of uniform cross section is very difficult because it requires the satisfaction of a boundary condition that contains an irrational term. However, the use of MACSYMA (a symbolic manipulation language developed at MIT and in use at DTNSRDC) makes it possible to solve such problems, since a larger number of terms can be retained and manipulated in various expansions without losing track of them in the enormously long and complex algebraic expressions.

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The general form of the boundary condition used in this method is developed in reference [4]. The special cases of reinforcement of a circular opening and a square-like opening are discussed in greater detail and actual numerical results are included.

This paper is a logical extension of the work described in reference [5] which dealt with the optimization of the shape of a class of unreinforced openings in large plates.

Mathematical Preliminaries

An opening of general shape in a large elastic plate of isotropic material of unit thickness is reinforced by a thin member of cross-sectional area A capable of withstanding axial forces only. Then, if the opening can be mapped into a unit circle by a function $z(\zeta)$, the equivalence of complex forces between those in the reinforcement and those in the plate at the opening boundary ($\zeta = \sigma$) is given by [4]

$$\phi(\sigma)+\frac{z(\sigma)}{\overline{z'(\sigma)}}\overline{\phi'(\sigma)}+\overline{\psi(\sigma)}=\sigma P\sqrt{\frac{z'(\sigma)}{\overline{z'(\sigma)}}}+C \qquad (1)$$

where $\phi(\sigma)$ and $\psi(\sigma)$ are the values of the functions $\phi(\zeta)$ and $\psi(\zeta)$ at the opening boundary, P is the axial force in the reinforcement, σ equals $e^{i\beta}$, and C is an arbitrary constant. Equivalence of tangential strain at the boundary requires

$$P=A\,(\sigma_\beta-\nu\sigma_\alpha) \qquad (2)$$

and the elastic equilibrium of a boundary element requires [4]

$$\sigma_\alpha=P\,\sqrt{z'\bar{z}'}$$

$$\tau_{\alpha\beta}=-\frac{1}{\sqrt{z'\bar{z}'}}\frac{\partial P}{\partial\beta} \qquad (3)$$

where the argument σ of the function z has been omitted for brevity and prime indicates differentiation with respect to σ.

The well-known relations between the stresses, σ_α, σ_β, and $\tau_{\alpha\beta}$ and the functions, $\phi(\zeta)$ and $\psi(\zeta)$, and equations (2) and (3) can be used to show that

$$P\sqrt{z'\bar{z}'}\,[A(1+\nu)+\sqrt{z'\bar{z}'}]=2A\,(\phi'\bar{z}'+\bar{\phi}'z') \qquad (4)$$

Equation (1) can be modified to

$$(\phi+\bar{\psi})\bar{z}'+z\bar{\phi}'=\sigma P\sqrt{z'\bar{z}'} \qquad (5)$$

where the argument σ has been omitted, for brevity, for the ϕ, ψ, and z functions; prime indicates differentiation with respect to the argument σ; and a bar (-) represents the complex conjugate of the function. Since P is a real quantity, it is possible to represent

$$P\sqrt{z'\bar{z}'}=\sum_{n=0}c_n\left(\sigma^n+\frac{1}{\sigma^n}\right) \qquad (6)$$

where the c_n are real. Functions $\phi(\zeta)$ and $\psi(\zeta)$ are known to have the following form:

$$\phi(\zeta)=S\zeta+\sum\frac{a_n}{\zeta^n}$$

$$\psi(\zeta)=D\zeta+\sum\frac{b_n}{\zeta^n} \qquad (7)$$

where

$$S=\frac{p+q}{4}$$

$$D=-\frac{p-q}{2}e^{-2i\theta}$$

and p and q are uniform stresses at infinity at an angle θ to the x and y-axes, respectively. The mapping function $z(\zeta)$ can be conveniently represented by