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A SIMPLE AND EFFECTIVE PIPE ELBOW ELEMENT—SOME NONLINEAR CAPABILITIES

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Abstract—In some earlier communications we presented the formulation of a simple pipe elbow element for linear analysis. In this paper we extend this formulation to include some nonlinear effects. Elastic–plastic conditions can be modeled, and some kinematic nonlinearities (due to large displacement beam behavior) can also be represented. The results of some sample solutions are given to illustrate the use of the element.

1. INTRODUCTION

In some recent papers we presented the formulation of a simple pipe elbow element[1–3]. In these communications we only considered the linear analysis of pipes, but it was already pointed out that a main objective in the development of the element was to obtain a simple and effective tool for nonlinear analysis of piping structures. More specifically, our objective was to provide an element that can be employed economically to model *assemblages* of pipe bends and straights for nonlinear static and dynamic analyses.

In principle, already a large number of different approaches are available for the nonlinear analysis of piping structures (see, for example, Ref. [4] for a survey of various approaches). Most notably finite element shell models can be employed to obtain very accurate solutions. However, the cost of analysis of a single pipe bend using these shell models can be very high and the analysis of assemblages of pipe bends and straights would be prohibitively expensive.

Apart from the costs of preparing and analyzing a refined shell model of a piping assemblage, it is generally recognized that frequently for the design of a piping structure the solution accuracy provided by such analysis is not necessary. For this reason considerable research efforts have been placed on the development of simplified analysis techniques. For linear analysis of piping structures these simplified methods—in very wide use—are based on simple curved beam theory and correction factors to estimate the flexibility of elbows and stress intensifications.

For more general nonlinear analysis a similar approach is hardly possible because the flexibilities and stresses of the structural components depend on the level and history of deformations, and distribution of plasticity, which requires that the complete stress distribution in a bend be predicted to sufficient accuracy. Also—as for all complex nonlinear static and dynamic analyses—the response solution must be calculated using an incremental formulation[5].

The basic aim in the development of our elbow element is to establish an element that is simple and computationally cost-effective and yet provides sufficiently accurate stress distributions to allow elastic–plastic analysis of pipe assemblages. The element should be theoretically sound and its formulation be well-understood and transparent as to the assumptions involved; the element should have good predictive capability but by its very nature will have limitations in its applicability.

Considering the analysis limitations of the element, it would be possible to extend the range of applicability, but such refinements must be chosen in a very careful manner and with considerable ingenuity so as not to unduly increase the complexity and cost of the element. Namely, with too complex an elbow element, the analysis of pipe bends may be performed more economically using shell elements, in particular when an element such as the DKT plate/shell element (with the Ilyushin yield criterion) is employed[6].

Based on the above thoughts our objectives in the development of the element can be summarized as follows:

- The element should be "easy in description and program input" to the analyst. Here, a beam-type formulation with input descriptions at centre-line nodal points only is clearly effective.
- The primary objective is to predict the flexibility of an elbow accurately whereas (in elastic–plastic analysis) stress distributions are only calculated as a means to reach that goal.
- The element should be computationally effective, i.e. give high accuracy stress predictions relative to the cost of analysis, and sufficiently accurate for a relatively wide range of analyses.
- The element should be able to model the interaction effects between elbows of different curvatures, elbows and straights and the effects of flanges on elbows. This is necessary because we want to use the element to analyse complete pipe assemblages.
- The theoretical formulation should be sound and transparent so that the assumptions involved are

clear. The element must satisfy all finite element convergence criteria, must not possess spurious zero energy modes and must be applicable to linear and nonlinear static and dynamic analyses.

In the next sections we summarize the basic formulation of our current elbow element. We then report on some solutions obtained with the element in nonlinear analysis. This paper describes our first developments on the element for nonlinear analysis, and we therefore conclude by summarizing some further developments that we are pursuing with the element.

2. SUMMARY OF BASIC ASSUMPTIONS

The formulation of the element for linear analysis was presented in earlier communications and we shall not repeat here that information. However, the basic kinematic assumptions for linear analysis are also those for nonlinear analysis and it is of value to summarize them in this presentation.

Figure 1 shows a pipe elbow element and the displacements and stresses to which we refer in the formulation.

2.1 *Finite element kinematic assumptions*

The elbow element is basically formulated as a curved beam of hollow circular section with the cross-section of the beam allowed to ovalize. The ovalization varies along the beam with full compatibility between elbows and straight sections. Using the notation and equations given in [1–3, 5], the basic equations describing the usual beam displacement behavior of the elbow in infinitesimal displacement analysis are:

$$u_i(r,s,t) = \sum_{k=1}^{4} h_k u_i^k + t \sum_{k=1}^{4} \bar{a}_k h_k V_{ti}^k + s \sum_{k=1}^{4} \bar{a}_k h_k V_{si}^k;\quad i = 1,2,3 \tag{1}$$

with

$$\mathbf{V}_t^k = \boldsymbol{\theta}^k \times {}^0\mathbf{V}_t^k;\quad \mathbf{V}_s^k = \boldsymbol{\theta}^k \times {}^0\mathbf{V}_s^k \tag{2}$$

where r, s, t = isoparametric coordinates; u_i = Cartesian displacements of material point (r, s, t); $h_k(r)$ = isoparametric interpolation functions; u_i^k = Cartesian displacements of nodal point k; $\boldsymbol{\theta}^k$ = rotations at nodal point k; $\bar{a}_k$ = outer radius of element at nodal point k; $\bar{a}_k = a + \delta/2$; δ = thickness of pipe skin; ${}^0V_{ti}^k$ = component i of unit vector ${}^0\mathbf{V}_t^k$, in direction t at nodal point k; and ${}^0V_{si}^k$ = component i of unit vector ${}^0\mathbf{V}_s^k$, in direction s at nodal point k.

The assumed displacements for ovalization are:

$$w_\xi(r,\phi) = \sum_{k=1}^{4} \left(\underbrace{\sum_{m=1}^{N_c} h_k c_m^k \sin 2m\phi}_{\text{in-plane bending}} + \underbrace{\sum_{m=1}^{N_d} h_k d_m^k \cos 2m\phi}_{\text{out-of-plane bending}} \right) \tag{3}$$

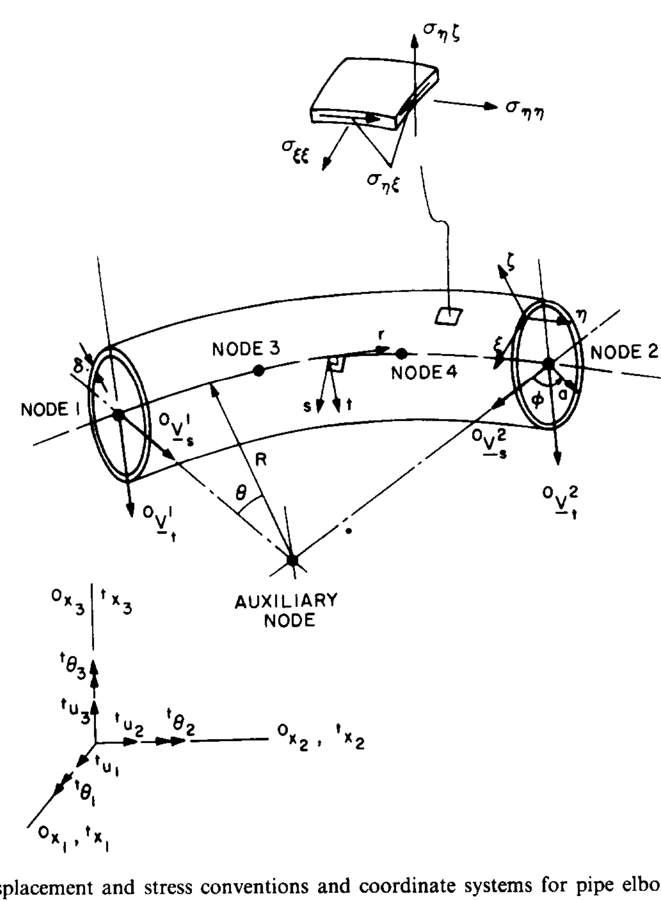


Fig. 1. Displacement and stress conventions and coordinate systems for pipe elbow elements.

where the assumption is that (see Ref. [1])

$$w_\zeta = -\frac{dw_\xi}{d\phi} \tag{4}$$

and w_ξ and w_ζ are the displacements of the pipe skin into the ξ and ζ directions, respectively. The c_m^k and d_m^k, $k = 1, 2, 3, 4$, are the unknown generalized ovalization displacements. Depending on the pipe geometry and the type of loading, it may be sufficient to include only the first, or first two, terms of one (or both) double summation(s). In the implementation of the element, we have allowed N_c to be 0 (no ovalization), 1, 2 or 3, and similarly for N_d. Without prior knowledge on how many ovalization degrees of freedom to include it is recommended to use N_c and/or N_d equal to 3. The total displacements of the element are the sum of the displacements presented in eqns (1) and (3). Hence, for example, with N_c and N_d equal to 3, a typical nodal point k carries the unknown displacements

$$\mathbf{u}^{kT} = [u_1^k\, u_2^k\, u_3^k\, \theta_1^k\, \theta_2^k\, \theta_3^k\, c_1^k\, c_2^k\, c_3^k\, d_1^k\, d_2^k\, d_3^k]. \tag{5}$$

We may note that the displacement assumptions given in eqns (1) and (3) correspond to a beam of circular cross-section that only distorts in its cross-sectional plane, hence warping is not accounted for. Also, we only allow the beam to ovalize in the specific patterns given in eqn (3).

Clearly, additional interpolations could be included to account for warping and more complex ovalization; however, then the element is bound to become computationally more expensive and when such refinement in the model is required, it is deemed more effective to use a full shell element idealization.

It is important to note that the cubic interpolation of *all* displacement components means that the rigid body mode criterion is directly satisfied and that the interpolations are very effective for a large displacement total Lagrangian formulation. Also, the fact that we use a cubic interpolation of displacements means that the element does not lock and reduced numerical integration is not necessary. We discuss the use of numerical integration in Section 3.3.

2.2 *Strain components*

With the displacement kinematic assumptions given, it is most important to identify the appropriate strain components for the element formulation. A study of Novozhilov's shell theory[7] for curved tubes shows that in linear analysis the important strain components for the elbow are the usual beam normal and shear strains and the additional strains due to the ovalization,

$$\epsilon_{\eta\eta} = \frac{w_\xi \sin\phi + \dfrac{dw_\xi}{d\phi}\cos\phi}{R - a\cos\phi} - \left[\left(\frac{1}{R - a\cos\phi}\right)^2 \frac{d^2 w_\zeta}{d\theta^2}\right]\zeta \tag{6}$$

$$\epsilon_{\xi\xi} = -\frac{1}{a^2}\left[w_\zeta + \frac{d^2 w_\zeta}{d\phi^2}\right]\zeta \tag{7}$$

and

$$\gamma_{\eta\xi} = \left(\frac{1}{R - a\cos\phi}\right)\frac{dw_\xi}{d\theta} \tag{8}$$

where the geometry and displacement variables are defined in Fig. 1.

Considering the above strain terms we recognize that the longitudinal strain $\epsilon_{\eta\eta}$ is calculated from the second derivative of the pipe skin radial displacement w_ζ. Hence, for convergence, continuity in $dw_\zeta/d\theta$ between adjacent elements has to be imposed and we enforce this continuity using a penalty procedure. The detailed algorithm for this task is presented in Ref. [2]. This algorithm is used to enforce the appropriate conditions on w_ζ between elbows of different curvatures, elbows and straight sections, and elbows and rigid flanges.

3. NONLINEAR FORMULATION

The basic equations used for the incremental formulation for non-linear analysis are given in Ref. (5). The formulation consists of the following kinematic and constitutive descriptions.

3.1 *Kinematics*

We have implemented the element for a total Lagrangian formulation with the assumptions that for the beam behavior the strain displacement matrices given in Ref. [5, p. 365] are used. Hence, the element accounts for large beam displacements and rotations but only small strains. However, for the ovalization action, we assume small displacements and use the strain terms given in eqns (6)–(8) corresponding to the initial configuration of the element.

The complete linear strain-displacement matrix for nodal point k of an element is thus

$${}_0^t\mathbf{B}_L^k \Big|_{\text{pipe}} = \begin{bmatrix} {}_0^t\mathbf{B}_L^k & \mathbf{B}_{0v1}^k & \mathbf{B}_{0v3}^k \\ \mathbf{0} & \mathbf{B}_{0v2}^k & \mathbf{B}_{0v4}^k \end{bmatrix} \tag{9}$$

where the superscript k stands for nodal point k, the matrix ${}_0^t\mathbf{B}_L^k$ is defined in Table 6.7 of Ref. (5) and the matrices $\mathbf{B}_{0v_i}^k$ accounting for ovalization are defined in Ref. [1].

The nonlinear strain-displacement matrix for the element is given by

$${}_0^t\mathbf{B}_{NL}^k \Big|_{\text{pipe}} = \begin{bmatrix} {}_0^t\mathbf{B}_{NL}^k & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{10}$$

where ${}_0^t\mathbf{B}_{NL}^k$ is given in Table 6.7 of Ref. (5).

The above displacement assumptions mean that the present element is only in a restricted sense applicable to large displacement analysis. Since deformation effects are not included in the strain-displacement terms corresponding to the ovalization degrees of freedom (in eqn 9) we have the following major assumptions:

- An increase (or decrease) in the flexibility of an element, as contributed by the ovalization, due to the change in the element curvature is neglected (the element radius of curvature R (see Fig. 1) is constant)

● Large displacement buckling/collapse in the ovalization modes cannot be calculated.

3.2 *Elasto-plasticity*

The elastic–plastic constitutive matrix is evaluated using flow theory as described in Ref. [5, p. 388], but only for the stress terms $\sigma_{\eta\eta}$, $\sigma_{\eta\xi}$, $\sigma_{\eta\zeta}$, $\sigma_{\xi\xi}$. Hence, plane stress conditions are assumed in the η–ξ plane. Also, the von Mises yield condition and isotropic strain hardening are assumed.

3.3 *Numerical integration*

For the evaluation of the element stiffness matrix and force vector that corresponds to the current element stresses numerical integration is used. Since the computational effort is directly proportional to the number of integration stations employed, it is important to use an effective scheme.

Although the element can be employed with a variable number of nodes (2, 3 or 4 nodes), in actual analysis it is usually most effective to employ the 4 node element, and eqn (1) has been written for that element. Figure 2 shows the integration schemes available for this element:

● Along the length of the element, 4 (or 3) point Gauss integration, or 5 (or 3) point Newton–Cotes integration.

● Around the circumference, 8, 12 or 24 point integration, composite trapezoidal formula.

● Through the thickness, 3, 5 or 7 point Newton–Cotes integration.

Considering the above integration schemes we note that for the integration along the element length the 4 point Gauss and 5 point Newton–Cotes formulas are usually effective. The advantage of Newton–Cotes over Gauss integration is that integration stations with positions at the element end-points are included, so that the development of plasticity in the element is in some analyses more accurately represented.

The number of integration stations required around the circumference depends on the number of ovalization modes used. To integrate accurately the strains corresponding to all ovalization modes in general three-dimensional analysis, the 24 point integration must be used, but for in-plane bending, the 12 point integration scheme is usually sufficient.

Considering the integration through the pipe skin thickness, 3 point integration is usually sufficient, but when the spread of plasticity through the pipe skin is important a higher order integration may need to be employed.

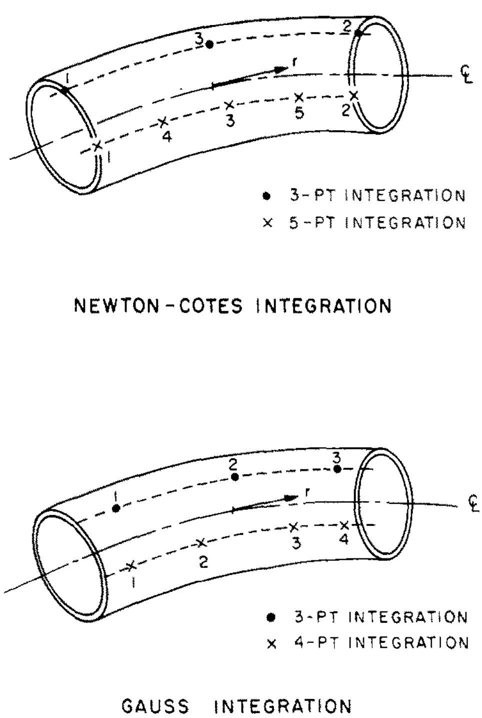
4. SAMPLE ANALYSES

The elbow element has been implemented in the ADINA program. The following two sample analyses illustrate the use of the elbow element in nonlinear analysis.

4.1 *Elastic–plastic analysis of Whatham pipe bend*

The 90° pipe bend with flanges at both ends, considered already in Ref. [2], was now analysed for its elastic–plastic response. This bend was earlier analysed by Whatham for its elastic response[8]. Figure 3 shows the pipe bend and the elbow element idealization used. Figure 4 shows the shell model of the same bend that was also solved in order to be able to evaluate the elbow analysis results. Large displacement effects are in this problem negligible so that materially-nonlinear-only analyses were carried out.

Figure 5 shows the moment vs rotation relationships as predicted using the two models for the



(a) Integration along the length of pipe elbow element

Fig. 2(a)

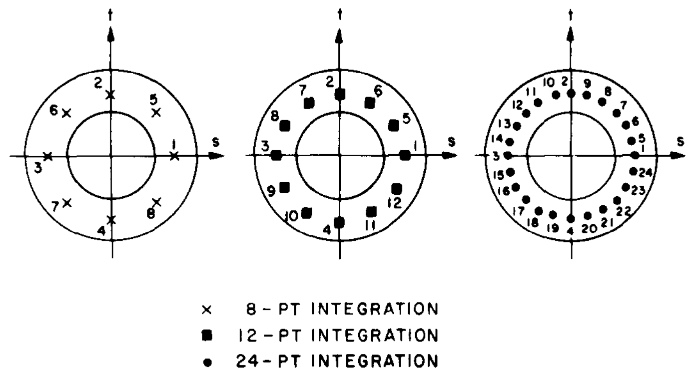


Fig. 2(b)

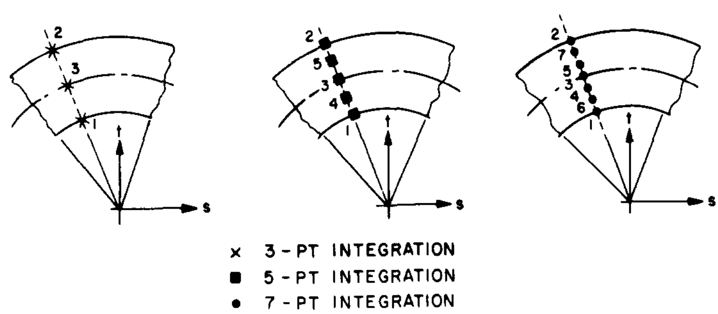


Fig. 2(c)

Fig. 2. Integration schemes used with 4-node pipe elbow element; for the exact locations of the Gauss integration points see Ref. [5], Table 5.3. (a) Integration along the length of pipe elbow element; (b) integration around the circumference of the pipe; (c) integration through the pipe skin.

bend. The predicted stiffness of the bend is slightly larger using the elbow idealization (partly because of the assumptions on the deformations of the elbow elements), but in fact good correspondence between the two solution results is noted. Also, only a relatively few load steps and equilibrium iterations per load step were required in the solutions.

Figure 6 shows the stresses as predicted in the nonlinear range using the two models. It is noted that while the calculated longitudinal stresses are in quite good agreement, the circumferential stress distributions show larger differences. On the other hand, at $M/M_0 = 2.5$ the bend has reached total collapse so that a close correspondence in the two predicted distributions can hardly be expected.

The important observation to be made is that the flexibility of the bend in the linear and nonlinear range is predicted quite accurately using the elbow element idealization.

4.2 *Nonlinear analysis of Sobel bend*

Figure 7 shows the piping structure that was analysed by Sobel and Newman[9] for nonlinear elastic–plastic response. Sobel and Newman used the

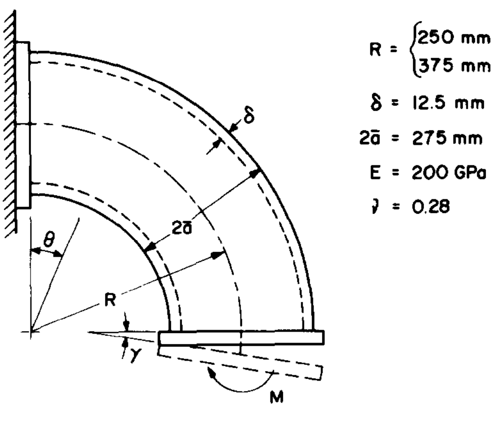


Fig. 3(a).

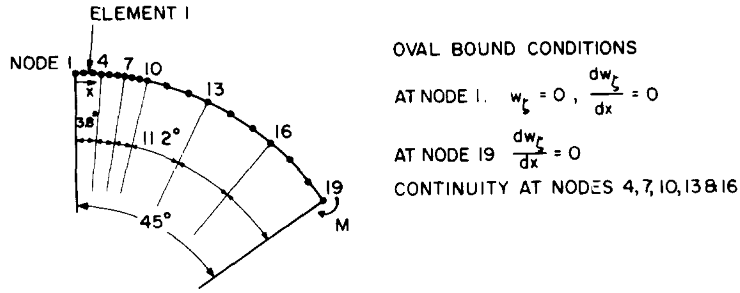


Fig. 3(b).

Fig. 3. Whatham pipe bend, E = Young's modulus, ν = Poisson's ratio. (a) Pipe bend considered; (b) elbow element model used (six 4-node elements with 5(thickn.) × 12(circum.) Newton–Cotes integr.).

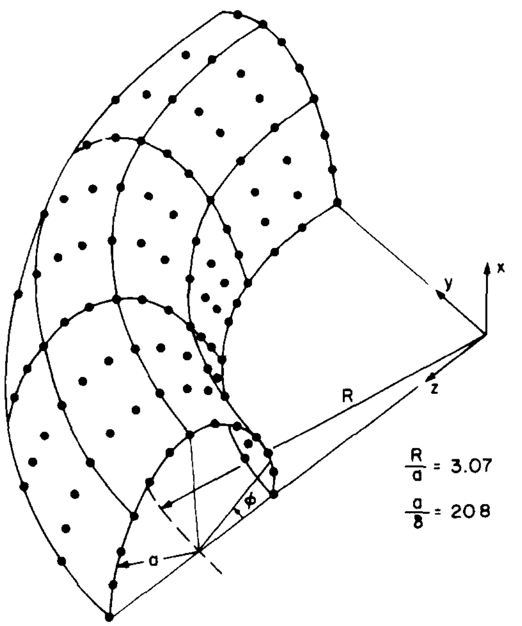


Fig. 4. Shell element model used for analysis of Whatham pipe bend, 4 × 4 × 4 Gauss integration.

Marc program and compared their response predictions with experimental results.

In our analysis we used the 5 element model shown in Fig. 7. Elements 1 and 5 are two Hermitian beam elements and elements 2, 3 and 4 are three elbow elements including ovalization. This model corresponds in many respects to the model of Sobel and Newman, who used 34 (evenly spaced) segments of the Marc element 17 around the hoop direction of the pipe bend, but also neglected the ovalization interaction effects between the straight and curved sections of the structure.

The material data used in our analysis are given in Fig. 7 and approximate the data given in Fig. 3 of Ref. (9).

Figure 8 shows the response predicted using the elbow model. It is seen that the predicted response follows very closely the experimental results up to about 6° of rotation. At larger values of rotation the predicted response lies above the experimentally measured moment values.

The difference between the predicted and the experimentally measured moments at larger rotations is probably, at least to some part, due to the fact that geometric nonlinearities in the ovalization displace-

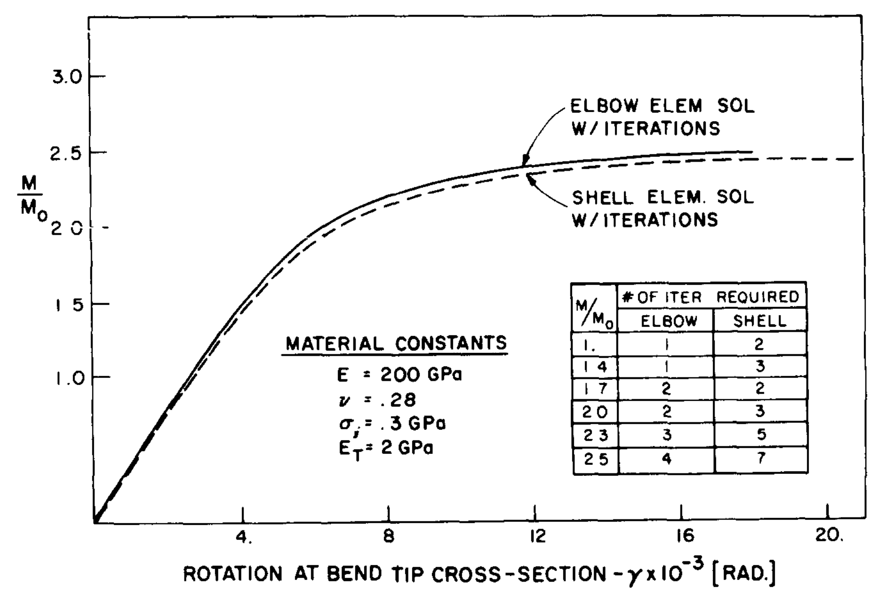


Fig. 5. Response of elbow and shell element models in analysis of Whatham pipe bend. M_0 is the limit load for yield initiation.

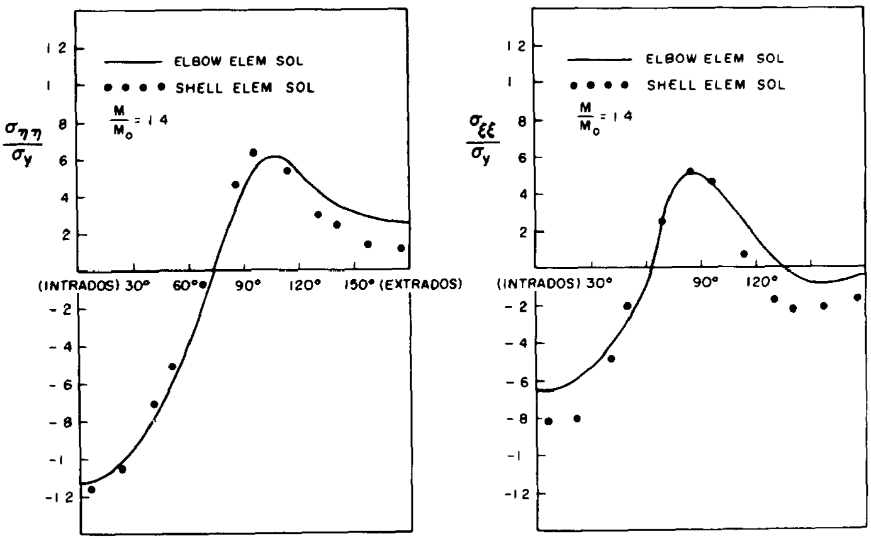


Fig. 6(a). Predicted longitudinal and circumferential stresses at $\theta = 45^\circ$ and at outside surface, $M/M_0 = 1.4$.

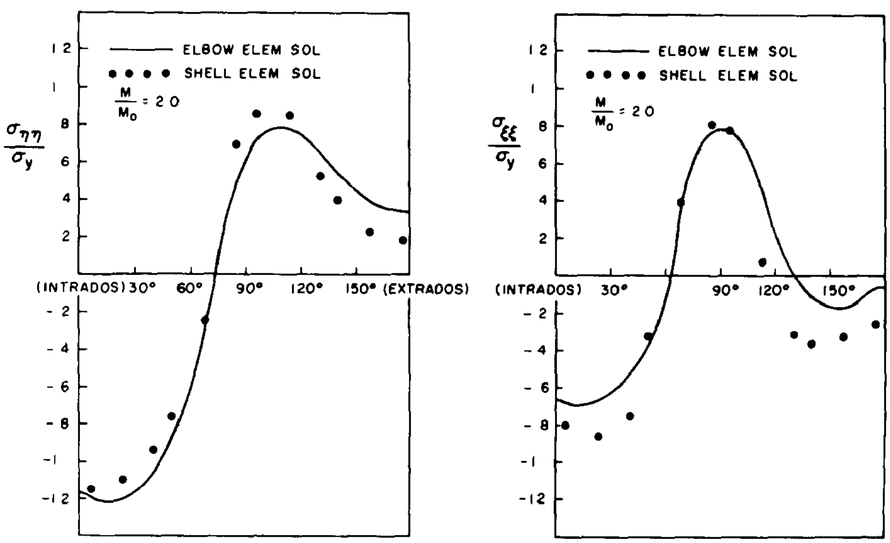


Fig. 6(b). Predicted longitudinal and circumferential stresses at $\theta = 45^\circ$ and at outside surface, $M/M_0 = 2.0$.

ments are not included in our model and the elastic–plastic material behavior was approximated.

5. CONCLUDING REMARKS

Although the elbow element discussed in the paper can already be effective for a variety of analyses, further research and development on the element is very desirable.

Considering the nonlinear capabilities, the assumptions on the kinematic nonlinearities summarized in Section 3.1 can be quite restrictive and the element formulation should be extended to include large displacement effects in the ovalization modes. Without this extension the element can only be employed when the large displacement membrane effects in the pipe skin are negligible.

To increase the applicability of the element, also additional inelastic material models should be implemented. Here, it may be effective to use force resultant yield criteria for the pipe skin, such as the Ilyushin yield condition[6], in order to avoid the numerical integration through the skin of the pipe.

Much of the effectiveness of the element depends on the numerical integration that is used, and that is required for an accurate response prediction. More experience need be gained with the different integration schemes already implemented and perhaps the

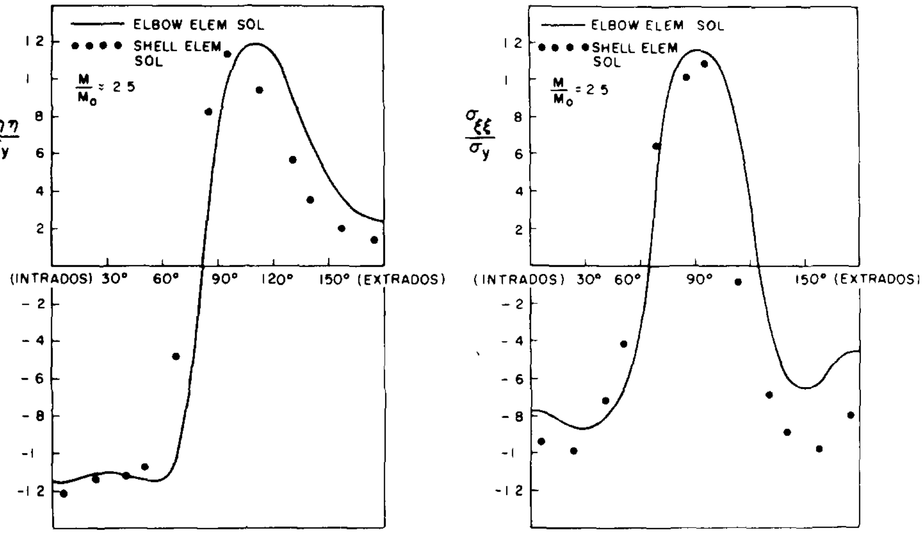


Fig. 6(c). Predicted longitudinal and circumferential stresses at $\theta = 45$ and at outside surface, $M/M_0 = 2.5$.

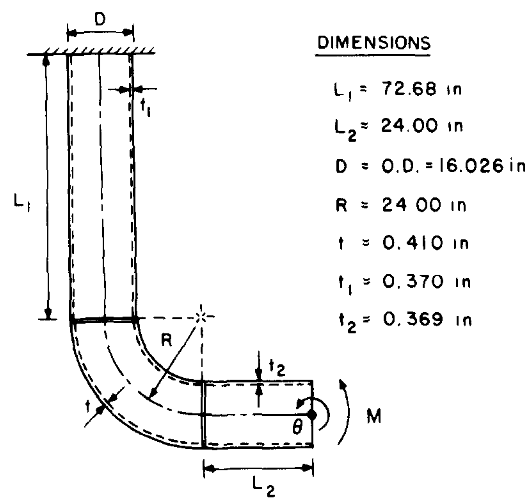


Fig. 7(a).

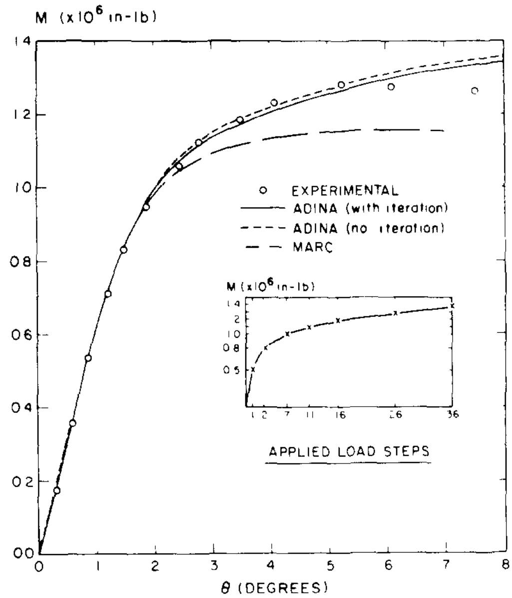


Fig. 8. Moment–rotation response predicted in analysis of pipe bend

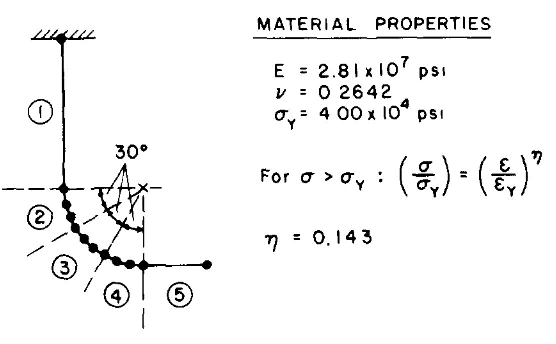


Fig. 7(b).

Fig. 7. Nonlinear analysis of pipe bend. (a) Structure; (b) finite element model, $5 \times 12 \times 3$ (long., circum., thickn.) Newton–Cotes integration.

use of additional integration methods should be explored.

The element has also been implemented for dynamic analysis, and a report on these capabilities is being prepared.

Finally, considerable effort is required to continue with the evaluation of the present capabilities of the element, and of all future enhancements, against experimental results and more refined numerical solutions.

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