An Iterative Finite Element Procedure for the Analysis of Piezoelectric Continua

PAOLO GAUDENZI* AND KLAUS-JÜRGEN BATHE
Massachusetts Institute of Technology, Cambridge, MA 02139

ABSTRACT: In this paper an iterative finite element procedure is presented for the analysis of two- and three-dimensional piezoelectric continua. The procedure is applied to the steady-state analysis of two-dimensional media subjected to mechanical and electrical loading.

INTRODUCTION

In recent years much effort has been directed to the research and development of new materials and associated new technologies for structural applications. A particularly prominent area of research has been the area of adaptive structure technology in which focus is given on the possibility of designing and building active structures with advanced distributed actuating and sensing capabilities (Wada, Fanson and Crawley, 1989). To this end, several types of materials such as magnetostrictive materials, piezoelectric and electrostrictive materials, and shape memory alloys are being tested in order to identify their possible use in engineering practice as part of an "intelligent structure".

Piezoelectric materials have been particularly taken into consideration for these kinds of applications because of their capability of producing in a relatively easy way both sensing and actuation.

Piezoelectric effects were discovered in 1880 by J. and P. Curie. When a mechanical force is applied to a piezoelectric material an electrical voltage is generated; this phenomenon is referred to as the direct piezoelectric effect. On the contrary, the converse piezoelectric effect is observed when the application of an electrical voltage produces strain in the material. In these cases an energy transfer is observed from mechanical to electrical energy or vice versa.

Piezoelectric materials have been widely used in transducers in several applications like strain gages, pressure transducers and accelerometers. The development of new products like lead zirconate titanate piezoelectric ceramics and low modulus piezoelectric films makes possible new kinds of structural engineering applications. In fact with these new piezoelectric materials it is possible to build composite structures in which the piezoelectric material is perfectly bonded (or even embedded in the case of fiber reinforced plastic composite laminates) to the passive (traditional) structure (Crawley and de Luis, 1987). In such an assembly both the actuating function (by means of the converse effect) and the sensing function (by means of the direct effect) can be performed, provided that appropriate locations and geometries are chosen for the piezoelectric part.

However, due to the rather complex nature of the physical behaviour, analytical and numerical methods for the analysis of piezoelectric materials are not yet fully developed.

One of the principal issues in the modeling is of course the coupling of the mechanical response with the electrical response; namely, the equations of mechanical equilibrium and continuity are coupled through the constitutive equations with the corresponding electrical equations. Also the physical behaviour of a piezoelectric material is frequently governed by a nonlinear constitutive equation so that a nonlinear mathematical model has to be established (Crawley and Lazarus, 1989).

For piezoelectric materials, a sound mathematical continuum model for the linear case has been established (Tiersten, 1969; Mindlin, 1972a; IEEE, 1987), and several analytical studies have used this model in mechanical vibrations of rods, plates and shells (Mindlin, 1972b).

Elements of a nonlinear theory have also been developed (Maugin, 1988). On the computational side, some finite element developments have been presented for the linear case (Tzou and Tseng, 1991; Lerch, 1990) based on the approach proposed by Allik and Hughes (Allik and Hughes, 1970). In this model the displacements and electrical potential are used as nodal unknowns. Recently, several numerical and analytical studies were performed on composite structures in which the piezoelectric part is bonded or embedded in a traditional structure or in a fiber reinforced plastic laminate (Crawley and de Luis, 1987; Im and Atluri, 1989; Wang and Rogers, 1991; Hagood and von Flotov, 1991; Ha, Keilers and Chang, 1992).

This paper describes a fairly simple finite element procedure that can be used to model the electro-mechanical coupled behavior of piezoelectric continua and that can also be used in a general nonlinear incremental finite element solution. The proposed technique is based on establishing

*Author to whom correspondence should be addressed.

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separately the finite element equations for the mechanical response and the electrical field. In this way the response for the converse and for the direct piezoelectric effects are solved. In order to fully account for the electro-mechanical coupling an iterative procedure is used. The procedure is demonstrated in the paper by the solution of some two-dimensional problems.

THE GOVERNING EQUATIONS OF PIEZOELECTRICITY

In this section the equations of the linear theory of piezoelectricity for the steady state case are briefly summarized (Tiersten, 1969).

Let us consider a piezoelectric body in three-dimensional space. The body occupies a region V bounded by a surface S with an outward unit normal vector with components n_i. The following equations have to be satisfied in V:

Mechanical equilibrium equations

$$\tau_{ij,j} + f_i^B = 0 \tag{1}$$

Strain-displacement relations

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2}$$

Maxwell's equations for the quasi-static electric field

$$D_{i,i} = 0 \tag{3}$$

$$E_i = -\phi_{,i} \tag{4}$$

Constitutive equations

$$\tau_{ij} = c_{ijkl}\epsilon_{kl} - e_{kij}E_k \tag{5}$$

$$D_i = e_{ikl}\epsilon_{kl} + \mathcal{E}_{ij}E_j \tag{6}$$

where u_i is the displacement vector, ϵ_{ij} the strain tensor, τ_{ij} the stress tensor, f_i^B the body force vector, ϕ the electric scalar potential, D_i the electric displacement vector, E_i the electric field vector, c_{ijkl} the elastic constitutive tensor, e_{kij} the piezoelectric tensor, $\mathcal{E}_{ij}$ the dielectric permittivity tensor. All the indices range over 1,2,3.

The boundary conditions are:

Natural mechanical boundary condition on S_f

$$\tau_{ij}n_j = f_i^S \tag{7}$$

Natural electrical boundary condition on S_σ

$$n_iD_i = \sigma^S \tag{8}$$

Essential mechanical boundary condition on S_u

$$u_i = u_i^* \tag{9}$$

Essential electrical boundary condition on S_ϕ

$$\phi = \phi^* \tag{10}$$

where f_i^S is the surface force and σ^S is the surface charge.

For the boundary surface S the following relations hold: $S_u \cup S_f = S$ and $S_u \cap S_f = 0$, $S_\phi \cup S_\sigma = S$ and $S_\phi \cap S_\sigma = 0$.

The constitutive equations are sometimes also written in the form

$$\epsilon_{ij} = F_{ijkl}\tau_{kl} + d_{kij}E_k \tag{11}$$

$$D_i = d_{ikl}\tau_{kl} + \mathcal{E}_{ij}E_j \tag{12}$$

where F_{ijkl} is the compliance tensor and d_{kij} is the piezoelectric (strain) tensor.

The following symmetries hold for the tensors that appear in the constitutive relations:

$$c_{ijkl} = c_{jikl} = c_{klij} \tag{13}$$

$$e_{kij} = e_{kji} \tag{14}$$

$$\mathcal{E}_{ij} = \mathcal{E}_{ji} \tag{15}$$

From Equation (3) it is clear that no body density charge σ^B is assumed to be present in the piezoelectric material.

The complete set of governing differential equations consists of 22 Equations (1)–(6) in 22 unknowns (u_i, ϵ_{ij}, τ_{ij}, ϕ, E_i, D_i), with the relevant boundary conditions (7)–(10).

A smaller set of four differential equations in four unknowns can be obtained from Equations (1)–(6) in terms of u_i and ϕ:

$$c_{ijkl}u_{k,lj} + f_i^B + e_{kij}\phi_{,kj} = 0 \tag{16}$$

$$e_{kij}u_{i,jk} - \mathcal{E}_{ij}\phi_{,ij} = 0 \tag{17}$$

Equation (16) is the three-dimensional equilibrium equation of the elastic body in terms of displacements with an additional term in which the piezoelectric tensor e_{kij} gives the electro-mechanical coupling. Equation (17) has the form of a divergence equation that describes field problems like heat flow or seepage in which, due to the coupling, a displacement dependent term is added.

If in Equation (16) we assume that the electric potential is given throughout the body, the problem can be solved in analogy with the usual approach for linear thermoelasticity, by considering a new body force vector f_i^{B*} instead of f_i^B, with

$$f_i^{B*} = f_i^B + e_{kij}\phi_{,kj} \tag{18}$$

A similar approach can be used for the electric field equation if u_i is assigned. The loading term in Equation (17) is equivalent to a body charge density σ^{B*}, with

$$\sigma^{B*} = -e_{kij}u_{i,jk} \tag{19}$$

THE FINITE ELEMENT EQUATIONS

Variational Principles

From Equations (1) and (7) that describe the three-dimensional mechanical equilibrium of the body in the field V and on the boundary S for each time t, the *principle of virtual work* can be derived.

For every time t and for every possible choice of virtual displacements δu_i that are zero at and corresponding to the essential mechanical boundary conditions (9), the following relation holds (Bathe, 1995):

$$\int_{{}^tV} {}^t\tau_{ij}\delta_t\epsilon_{ij}\,d^tV = \int_{{}^tV} {}^tf_i^B\delta u_i\,d^tV + \int_{{}^tS_f} {}^tf_i^S\delta u_i^S\,d^tS_f \tag{20}$$

where ${}^t\tau_{ij}$ is the stress at time t, $\delta_t\epsilon_{ij} = [1/2(\partial\delta u_i/\partial^t x_j + \partial\delta u_j/\partial^t x_i)]$ is the virtual strain corresponding to δu_i, ${}^tf_i^B$ and ${}^tf_i^S$ are the body forces and the surface tractions at time t, ${}^tV$ and ${}^tS_f$ are the volume occupied at time t and the corresponding surface on which the tractions ${}^tf_i^S$ are prescribed and δu_i^S is the virtual displacements on ${}^tS_f$.

In analogy with the principle of virtual work a *principle of virtual electric potentials* can be stated as follows from Equations (3) and (7). For every time t and for every choice of virtual electric potential $\delta\phi$ that is zero at and corresponding to the essential electrical boundary condition (10), the following relation holds:

$$\int_{{}^tV} {}^tD_i\delta_tE_i\,d^tV = -\int_{{}^tS_\sigma} {}^t\sigma^S\delta\phi^S d^tS_\sigma \tag{21}$$

where ${}^tD_i$ is the electric displacement vector at time t, $\delta_tE_i = -(\partial\delta\phi/\partial^t x_i)$ is the virtual electric field corresponding to $\delta\phi$, ${}^t\sigma^S$ is the surface charge density, and $\delta\phi^S$ is the virtual electric potential on the boundary surface ${}^tS_\sigma$.

In these variational principles no restrictions on constitutive relations are present (Bathe, 1995); consequently they can be applied also for the case of material nonlinearity. In Equations (20) and (21) in which no dynamic effect is considered, time is used as a convenient variable which denotes different intensities of load applications and correspondingly different configurations.

The Case of Linear Constitutive Relations

We now restrict the analysis to the case of linear constitutive relations. If we substitute Equations (5) and (6) into Equations (20) and (21) the following set of equations is obtained:

$$\int_V c_{ijkl}\epsilon_{kl}\delta\epsilon_{ij}\,dV - \int_V e_{kij}E_k\delta\epsilon_{ij}\,dV = \int_V f_i^B\delta u_i\,dV + \int_{S_f} f_i^S\,\delta u_i^S\,dS_f \tag{22}$$

$$\int_V e_{ikl}\epsilon_{kl}\delta E_i\,dV + \int_V \mathcal{E}_{ij}E_j\,\delta E_i\,dV = -\int_{S_\sigma}\sigma^S\delta\phi^S dS_\sigma \tag{23}$$

The same equations can be written also in matrix form:

$$\int_V \underline{\bar\epsilon}^T\underline{C}\,\underline{\epsilon}\,dV - \int_V \underline{\bar\epsilon}^T\underline{e}\,\underline{E}\,dV = \int_V \underline{\bar u}^T\underline{f}^B dV + \int_{S_f}\underline{\bar u}^{ST}\underline{f}^S dS_f \tag{24}$$

$$\int_V \underline{\bar E}^T\underline{e}\,\underline{\epsilon}\,dV + \int_V \underline{\bar E}^T\underline{\mathcal{E}}\,\underline{E}\,dV = -\int_{S_\sigma}\sigma^S\bar{\underline{\phi}}^S dS_\sigma \tag{25}$$

Finite Element Discretization of the Linear Piezoelectric Equations

We now formulate the generic finite element equations for the variables $\underline{u}$ and ϕ. For every finite element m of the considered body we assume

$$\underline{u}^{(m)} = \underline{H}_u^{(m)}\hat{\underline{u}} \quad \phi^{(m)} = \underline{H}_\phi^{(m)}\underline{\phi} \tag{26}$$

where $\hat{\underline{u}}$ is the vector of nodal displacement and $\underline{\phi}$ is the vector of electric potential of the discretized body. By substituting Equations (26) into Equations (2) and (4) we obtain

$$\underline{\epsilon}^{(m)} = \underline{B}_u^{(m)}\hat{\underline{u}} \quad \underline{E}^{(m)} = \underline{B}_\phi^{(m)}\underline{\phi} \tag{27}$$

We can then obtain, following the classical procedure of summing over all the elements:

$$\sum_m\int_{V^{(m)}}\underline{B}_u^{(m)T}\underline{C}\,\underline{B}_u^{(m)}dV^{(m)}\hat{\underline{u}} - \sum_m\int_{V^{(m)}}\underline{B}_u^{(m)T}\underline{e}\,\underline{B}_\phi^{(m)}dV^{(m)}\underline{\phi} = \sum_m\int_{V^{(m)}}\underline{H}_u^{(m)T}\underline{f}^{B(m)}dV^{(m)} + \sum_m\int_{S_f^{(m)}}\underline{H}_u^{S(m)T}\underline{f}^{S(m)}dS_f^{(m)} \tag{28}$$

$$-\sum_m\int_{V^{(m)}}\underline{B}_\phi^{(m)T}\underline{e}\,\underline{B}_u^{(m)}dV\,\hat{\underline{u}} - \sum_m\int_{V^{(m)}}\underline{B}_\phi^{(m)T}\underline{\mathcal{E}}\,\underline{B}_\phi^{(m)}dV\,\underline{\phi}$$

$$= \sum_m \int_{S_\sigma^{(m)}} \underline{H}_\sigma^{S(m)T} \sigma^{S(m)} dS_\sigma^{(m)} \qquad (29)$$

These equations can also be written in a more compact form as follows:

$$\underline{k}_{uu}\hat{\underline{u}} + \underline{k}_{u\phi}\underline{\phi} = \underline{F}_u^B + \underline{F}_u^S \qquad (30)$$

$$\underline{k}_{\phi u}\hat{\underline{u}} + \underline{k}_{\phi\phi}\underline{\phi} = \underline{F}_\phi^S \qquad (31)$$

with:

$$\underline{k}_{uu} = \sum_m \int_{V^{(m)}} \underline{B}_u^{(m)T}\underline{C}\underline{B}_u^{(m)} dV^{(m)}$$

$$\underline{k}_{u\phi} = \underline{k}_{\phi u}^T = -\sum_m \int_{V^{(m)}} \underline{B}_u^{(m)T}\underline{e}\underline{B}_\phi^{(m)} dV^{(m)}$$

$$\underline{k}_{\phi\phi} = -\sum_m \int_{V^{(m)}} \underline{B}_\phi^{(m)T}\underline{\varepsilon}\underline{B}_\phi^{(m)} dV^{(m)}$$

$$\underline{F}_u^B = \sum_m \int_{V^{(m)}} \underline{H}_u^{(m)T}\underline{f}^{B(m)} dV^{(m)}$$

$$\underline{F}_u^S = \sum_m \int_{S_f^{(m)}} \underline{H}_u^{S(m)T}\underline{f}^{S(m)} dS_f^{(m)}$$

$$\underline{F}_\phi^S = \sum_m \int_{S_\sigma^{(m)}} \underline{H}_\phi^{S(m)T}\sigma^{S(m)} dS_\sigma^{(m)}$$

where $\underline{k}_{uu}$ is the mechanical stiffness matrix, $\underline{k}_{\phi\phi}$ is the electrical permittivity matrix, $\underline{k}_{u\phi}$ is the piezoelectric matrix, $\underline{F}_u^B$ is the body force loading vector, $\underline{F}_u^S$ is the surface force loading vector, $\underline{F}_\phi^S$ is the surface density charge vector.

An Iterative Approach for Solution

Equations (30) and (31) are, respectively, the finite element expressions of the converse and the direct piezoelectric effects. If we assume that in Equation (30) the potential is given throughout the body, we can move the expression $\underline{k}_{u\phi}\underline{\phi}$ to the right hand side and consider it as an additional load vector. In this way the solution of Equation (30) gives:

$$\hat{\underline{u}} = \underline{k}_{uu}^{-1}(\underline{F}_u^B + \underline{F}_u^S - \underline{k}_{u\phi}\underline{\phi})$$

Here the influence of the displacement field on the electric field has been neglected, but the problem has now a much simpler formulation.

The same process can be applied to Equation (31) as far as the displacement contribution is concerned. If the displacement field is known, an additional loading vector can be obtained moving $\underline{k}_{\phi u}\hat{\underline{u}}$ to the r.h.s. of the equation. The solution of the problem can be written as:

$$\underline{\phi} = \underline{k}_{\phi\phi}^{-1}(\underline{F}_\phi^S - \underline{k}_{\phi u}\hat{\underline{u}})$$

Here the effect of the electric field on the displacement field has been neglected.

In some cases it is possible to deal with the direct and converse effects in a separate way but in general the coupling between Equations (30) and (31) has to be fully taken into account. A possible way to obtain the fully coupled solution is to solve simultaneously Equations (30) and (31) for $\underline{u}$ and $\underline{\phi}$. Alternatively, an iterative procedure can be adopted. Its steps can be briefly outlined as follows:

1. Solve Equation (31) assuming that $\hat{\underline{u}} = 0$, thus obtain $\underline{\phi}$, $\underline{E}$, and $\underline{D}$;
2. Substitute the obtained value for $\underline{\phi}$ into Equation (30) and solve for $\hat{\underline{u}}$, $\underline{\epsilon}$, $\underline{\tau}$;
3. Substitute the value obtained for $\hat{\underline{u}}$ into Equation (31); again solve for $\underline{\phi}$, $\underline{E}$, and $\underline{D}$.
4. Compare the values of $\underline{\phi}$ obtained in step 3 with those obtained in step 1, by evaluating whether the following tolerance condition is satisfied:

$$(\| \underline{\phi}^{(3)} - \underline{\phi}^{(1)} \|)/\| \underline{\phi}^{(1)} \| \le \beta_1$$

   where β_1 is a tolerance parameter;
5. Substitute again the value obtained for $\underline{\phi}$, into Equation (30) and solve for $\hat{\underline{u}}$, $\underline{\epsilon}$, $\underline{\tau}$;
6. Compare the values of $\underline{u}$ obtained in step 5 with those obtained in step 2, by evaluating whether the following tolerance condition is satisfied:

$$(\| \hat{\underline{u}}^{(5)} - \hat{\underline{u}}^{(2)} \|)/\| \hat{\underline{u}}^{(2)} \| \le \beta_2$$

   where β_2 is a tolerance parameter. If the above two convergence conditions are not fulfilled continue with the analysis, repeating steps 3 to 6, until the two tolerance conditions are fulfilled.

A key advantage of this iterative approach is that already existing finite element programs that solve classical solid mechanics problems and field problems, like heat transfer, can directly be used in a reliable way providing that the constitutive law is modified.

This way both geometric and material nonlinearities can directly be included, and also a reduction in solution time may be accomplished. Of course the solution procedure assumes that convergence is reached in a reasonable number of iterations.

The Case of Nonlinear Constitutive Relations

The relations (20) and (21) express respectively the mechanical and electrical equilibrium at all times of interest. In order to establish a general solution scheme for nonlinear problems the development of incremental equilibrium equations is necessary.

We aim to establish a procedure that is both iterative, in the spirit of the discussion in the section, "An Iterative Approach for Solution," and also incremental, dealing at every iteration with the mechanical or electrical equilibrum.

Let us consider again Equation (20). We assume that the conditions at time t have been calculated and that the displacements are to be determined for time $t + \Delta t$, where Δt is the time increment (note that in the steady-state case time is a dummy variable). As far as electrical variables are concerned, we use the last calculated values, that is to say, the mechanical displacements are the only primary unknown variables. Mechanical equilibrium is considered at time $t + \Delta t$ in order to solve for the displacements at time $t + \Delta t$:

$$\int_{{}^tV} C_{ijkl}\epsilon_{kl}\delta\epsilon_{ij}\,dV = {}^{t+\Delta t}R - \int_{{}^tV} {}^t\tau_{ij}\,\delta\epsilon_{ij}\,dV \qquad (32)$$

where

$${}^{t+\Delta t}R = \int_{{}^tV} {}^{t+\Delta t}f_i^B\,\delta u_i\,dV + \int_{{}^tS_f} {}^{t+\Delta t}f_i^S\,\delta u_i^S\,dS_f \qquad (33)$$

In Equation (32) ${}^t\tau_{ij}$ includes the last calculated value for the electric field, see Equation (5). Equation (32) represents a linearization of the mechanical response and the first step of a Newton-Raphson iteration (Bathe, 1995).

Whereas the proposed expression for the mechanical equilibrium equation allows to calculate the displacements for every time step, assuming that the value of the electric field is known, a similar expression can be written for the electrical incremental equilibrium, assuming that the stresses are known.

To that purpose let us consider now Equation (21). We assume that the conditions at time t have been calculated and that the electrical displacements are to be determined for time $t + \Delta t$, where Δt is the time increment. As far as the mechanical variables are concerned, we use the last calculated values, so that the electric potential is the only unknown variable. Electrical equilibrium is considered at time $t + \Delta t$ in order to solve for the electric potential at time $t + \Delta t$:

$$\int_{{}^tV} \varepsilon_{ij}E_j\,\delta E_i\,dV = -\int_{{}^tS_\sigma} {}^{t+\Delta t}\sigma^S\,\delta\phi^S\,dS_\sigma - \int_{{}^tV} {}^tD_i\,\delta E_i\,dV \qquad (34)$$

In Equation (34) ${}^tD_i$ includes the last calculated mechanical effect, see Equation (6).

In order to obtain a fully coupled solution of Equation (32) and Equation (34) we proceed as follows: we first perform Newton-Raphson iterations on Equation (32) using the last calculated value for the electric field in the calculation of the stresses ${}^{t+\Delta t}\tau_{ij}^{(k)}$ until at the k-th iteration

$${}^{t+\Delta t}R - \int_{{}^tV} {}^{t+\Delta t}\tau_{ij}{}^{(k)}\delta\epsilon_{ij}\,dV = 0 \qquad (35)$$

We then move to Equation (34) and we use the last calculated value the mechanical effect in the calculation of the electrical displacements and we perform Newton-Raphson iterations on Equation (34). At this stage an updated value for ${}^{t+\Delta t}E_k$ is available for solving again Equation (32). This procedure is continued until convergence is reached for both the mechanical equilibrium and the electrical equilibrium at time $t + \Delta t$.

NUMERICAL RESULTS

In this section the results of two simple electroelastic analyses for the linear case and for the case of nonlinear constitutive equations are given in order to demonstrate the capability of performing a coupled electro-mechanical analysis of 2D continua using the procedure described in the section, "The Finite Element Equations".

Analysis of a Two-Dimensional Beam under Electrical and Mechanical Loading

Consider a rectangular strip of piezoelectric material occupying the region $|x| \leq l$, $|z| \leq h$ of a two-dimensional space, as shown in Figure 1. The material has been polar-

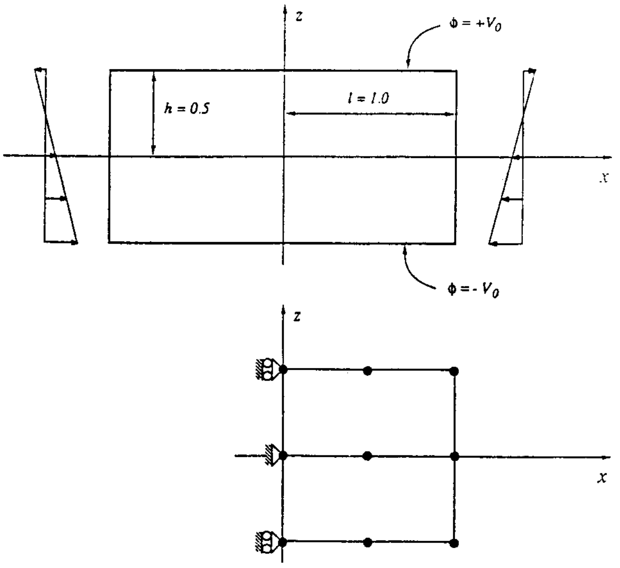

Figure 1.

Table 1. Data used in solution of problem in Figure 1.

s_{11}	0.1944444 E-4 mm^2/Newton
s_{13}	−0.083333 E-4 mm^2/Newton
σ_0	−5.0 Newton/mm^2
σ_1	2.0 E + 1 Newton/mm^3
V_0	1.0 E + 3 Volt
h	5.0 E-1 mm
d_{31}	−1.8 E-7 mm/Volt
d_{33}	3.6 E-7 mm/Volt
$\mathscr{E}_{33}$	1.505 E-8 Newton/Volt^2

ized along the thickness, that is along the z direction, and is assumed to be transversely isotropic.

The governing equations of the two-dimensional plane-stress problem are given in Appendix 1.

The following boundary conditions apply for the problem:

At $z = \pm h$

$$\phi = \pm V_0 \qquad \sigma_z = 0 \qquad \tau_{xz} = 0$$

At $x = \pm l$

$$D_x = 0 \qquad \sigma_x = \sigma_0 + \sigma_1 z \qquad \tau_{xz} = 0$$

The analytical solution of the problem is given in Appendix 1.

Only one half of the structure is considered in the finite element modeling due to symmetry with respect to the z axis and the data in Table 1 are used. Both electrical and mechanical simulations use a single 9 node element. The iterative procedure converges to the exact solution in five steps, since the values computed for stresses at step 5 are equal to the values calculated at step 2. Some calculated values for displacements, stresses, electrical potential, and electrical displacements at the different steps are reported in Table 2. In the same table the exact solution is also given.

Analysis of the Piezoelectric Actuation of an Aluminum Structure

We now consider the experiments performed on an aluminum cantilever by Anderson and Crawley (Anderson and Crawley, 1989). Two 2024 aluminum beam specimen are examined. They have thicknesses of 3.21 mm and 1.59 mm, each has a length of 356 mm and width of 51 mm. Two G-1195 piezoceramic actuators with dimensions 63.5 × 25.4 × 0.254 mm were bonded to each of the upper and lower surfaces of the beam at a distance of 25.4 mm from the clamped end of the cantilever. A small strip was left free in order to enable strain measurements.

The beam specimen were statically deformed by applying an electric field in a direction normal to the middle plane of

Table 2. Convergence of displacements, electric potentials, electric displacements.

x	z	w_{exact}	$w^{(2)}$	$w^{(5)}$	$w^{(2)}/w_{ex}$	$w^{(5)}/w_{ex}$
1.0	0.0	−1.72916 E-4	−1.94444 E-4	−1.72916 E-4	1.1244	1.0000
1.0	0.5	−5.22152 E-4	−5.54444 E-4	−5.22152 E-4	1.0618	1.0000
1.0	−0.5	1.56181 E-4	1.23889 E-4	1.56181 E-4	0.7932	1.0000
0.5	0.0	−0.43229 E-4	−0.48611 E-4	−0.43229 E-4	1.1244	1.0000
0.5	0.5	−3.92464 E-4	−4.08611 E-4	−3.92465 E-4	1.0411	1.0000
0.5	−0.5	2.85868 E-4	2.69722 E-4	2.85868 E-4	0.9435	1.0000
		u_{exact}	$u^{(2)}$	$u^{(5)}$	$u^{(2)}/u_{ex}$	$u^{(5)}/u_{ex}$
1.0	0.0	2.62778 E-4	2.62778 E-4	2.62778 E-4	1.0000	1.0000
1.0	0.5	4.35694 E-4	4.57222 E-4	4.35694 E-4	1.0494	1.0000
1.0	−0.5	0.89861 E-4	0.68333 E-4	0.89861 E-4	0.7604	1.0000
0.5	0.0	1.31389 E-4	1.31389 E-4	1.31389 E-4	1.0000	1.0000
0.5	0.5	2.17847 E-4	2.28611 E-4	2.17847 E-4	1.0494	1.0000
0.5	−0.5	0.44930 E-4	0.34166 E-4	0.44930 E-4	0.7604	1.0000
		ϕ^{exact}	$\phi^{(1)}$	$\phi^{(3)}$	$\phi^{(1)}/\phi_{ex}$	$\phi^{(3)}/\phi_{ex}$
1.0	0.0	29.9003	0.0	29.9004 E-4	0.0	1.0000
x	z	$\sigma_{xx\ exact}$	$\sigma_{xx}^{(2)}$	$\sigma_{xx}^{(5)}$	$\sigma_{xx}^{(2)}/\sigma_{xx\ ex}$	$\sigma_{xx}^{(5)}/\sigma_{xx\ ex}$
0.887298	−0.387298	−12.745966	−12.7460	−12.7460	1.0000	1.0000
0.887298	0.0	−5.0	−5.0	−5.0	1.0000	1.0000
0.887298	+0.387298	2.745966	2.74597	2.74597	1.0000	1.0000
		$D_{z\ exact}$	$D_z^{(1)}$	$D_z^{(3)}$	$D_z^{(1)}/D_{z\ ex}$	$D_z^{(3)}/D_{z\ ex}$
0.887298	−0.387298	−0.292 E-4	−0.301 E-4	−0.292 E-4	1.0308	1.0000
0.887298	0.0	−0.292 E-4	−0.301 E-4	−0.292 E-4	1.0308	1.0000
0.887298	+0.387298	−0.292 E-4	−0.301 E-4	−0.292 E-4	1.0308	1.0000

Table 3. Comparison between experimental and numerical results (ϵ_{11} is evaluated at P; see Figure 3).

Thickness mm	ϕ Volt	E_3 Volt/mm	Microstrain Experimental	Microstrain Numerical
3.21	50	197.9	14.7	15.54
	100	393.7	32.6	32.48
	150	590.6	52.4	50.90
1.59	50	197.9	26.8	24.26
	100	393.7	60.1	52.03

the actuators. In order to produce bending the actuators on the top of the beam were subjected to a field equal in value but opposite in sign with respect to the actuators on the bottom. During the static deformation different levels of voltages were applied, as reported in Table 3. For the corresponding level of the electric field a nonlinear constitutive relation in the piezoelectric coupling was shown to be present (Crawley and Lazarus, 1989). In Figure 2 the experimental curve obtained for the in-plane strain versus the normal component of the electric field of a free piezoceramic specimen is shown (Crawley and Lazarus, 1989). In the same figure the nonlinear constitutive relation used in the numerical simulation is also shown. A quadratic interpolation has been used to model the experimental curve (Anderson and Crawley, 1989).

A two-dimensional finite element analysis was performed in the bending plane using 2D elements both to represent the aluminum structure and the actuators. A portion of the mesh close to the end of the actuators is shown in Figure 4(a). Perfect bond was assumed between the actuators and the structure.

Table 3 shows a comparison between the numerical prediction obtained with 10 time steps and the experimental results for the bending strain measured at point P (see Figure 3 on the top of the structure). Different loading cases are reported for the two considered thickness. In Figure 4(b) the distribution of in-plane strain along the thickness is reported for the case of beam thickness 1.59 mm and applied potential ϕ = 100 volts.

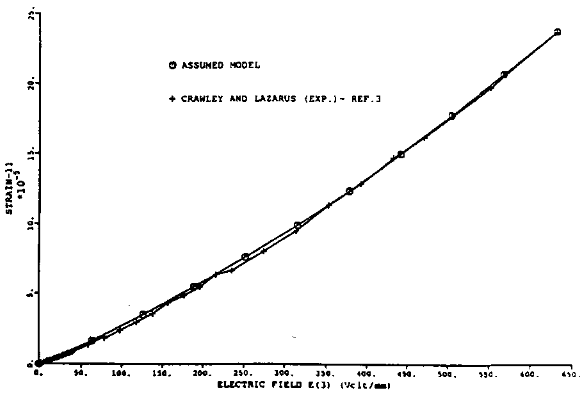

Figure 2.

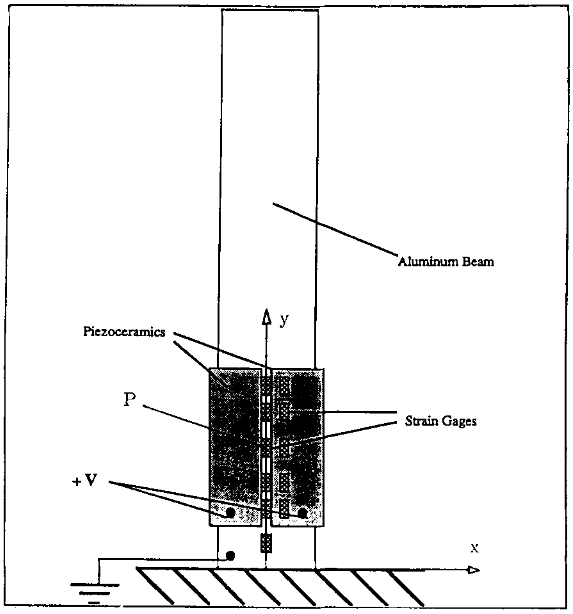

Figure 3.

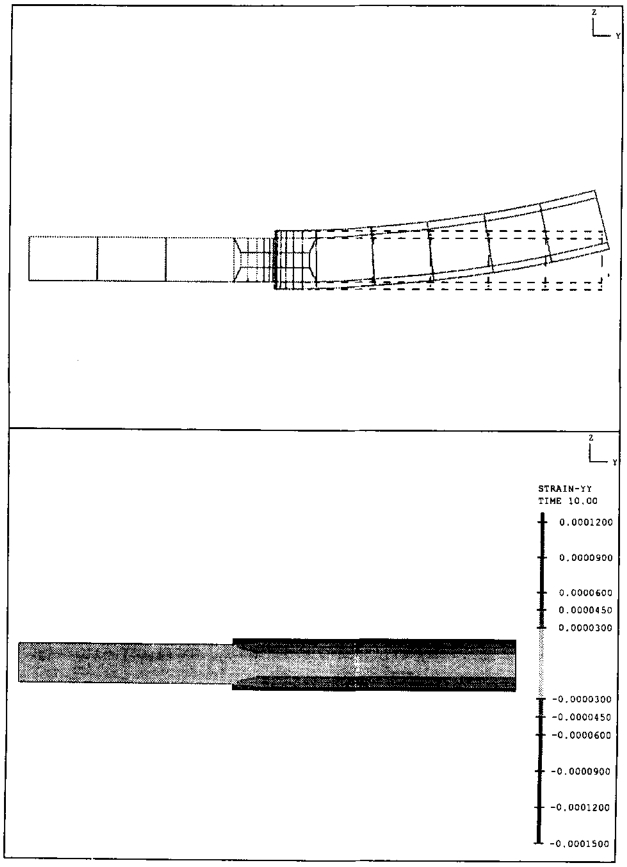

Figure 4.

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APPENDIX 1: A 2D CLOSED FORM SOLUTION

Consider a rectangular strip of piezoceramics occupying the region $|x| \leq l$, $|z| \leq h$ of a two-dimensional space, as shown in Figure 1. The piezoelectric material has been polarized along the thickness, that is along z direction.

The governing equations can be written as follows (Parton et al., 1989):

$$D_{x,x} + D_{z,z} = 0 \tag{36}$$

$$\sigma_{x,x} + \tau_{xz,z} = 0 \qquad \tau_{xz,x} + \sigma_{z,z} = 0 \tag{37}$$

$$E_x = -\frac{\partial \phi}{\partial x} \qquad E_z = -\frac{\partial \phi}{\partial z} \tag{38}$$

$$\epsilon_x = \frac{\partial u}{\partial x} \qquad \epsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{39}$$

$$D_x = \mathcal{E}_{11} E_x + d_{15} \tau_{xz} \quad D_z = \mathcal{E}_{33} E_z + d_{31}\sigma_x + d_{33}\sigma_z \tag{40}$$

$$\epsilon_x = s_{11}\sigma_x + s_{13}\sigma_z + d_{31}E_z \qquad \epsilon_z = s_{13}\sigma_x + s_{33}\sigma_z + d_{33}E_z$$

$$\gamma_{xz} = s_{55}\tau_{xz} + d_{15}E_x \tag{41}$$

Here Equations (40) and (41) are the constitutive relations derived for the plane stress case in the form of Equations (11) and (12) of section 2.

Consider now the following boundary conditions:

At $z = \pm h$

$$\phi = \pm V_0 \qquad \sigma_z = 0 \qquad \tau_{xz} = 0 \tag{42}$$

At $x = \pm l$

$$D_x = 0 \qquad \sigma_x = \sigma_0 + \sigma_1 z \qquad \tau_{xz} = 0 \tag{43}$$

For this problem the exact solution was given by Boriseiko et al. (Parton et al., 1989):

$$u = s_{11}\left(\sigma_0 - \frac{d_{31}V_0}{s_{11}h}\right)x + s_{11}(1 - k_{31}^2)\sigma_1 xz \tag{44}$$

$$w = s_{13}\left(\sigma_0 - \frac{d_{33}V_0}{s_{13}h}\right)z + s_{13}(1 - k_s^2)\sigma_1\frac{z^2}{2} - s_{11}(1 - k_{31}^2)\sigma_1\frac{x^2}{2} \tag{45}$$

$$\phi = V_0\frac{z}{h} - \frac{d_{31}\sigma_1}{2\mathcal{E}_{33}}(h^2 - z^2)$$

$$E_z = -\frac{V_0}{h} - \frac{d_{31}\sigma_1}{\mathcal{E}_{33}}z \qquad D_z = -\mathcal{E}_{33}\frac{V_0}{h} + d_{31}\sigma_0 \tag{46}$$

$$\sigma_z = \tau_{xz} = E_x = D_x = 0 \tag{47}$$

with

$$k_{31}^2 = \frac{d_{31}^2}{s_{11}\mathcal{E}_{33}} \qquad k_s^2 = \frac{d_{33}d_{31}}{s_{13}\mathcal{E}_{33}} \tag{48}$$

REFERENCES

Allik, H. and T. J. R. Hughes. 1970. "Finite Element Method for Piezoelectric Vibration", *International Journal for Numerical Methods in Engineering*, 2:151–157.

Anderson, E. H. and E. F. Crawley. 1989. *Piezoceramic Actuation of One- and Two-Dimensional Structures*, Cambridge, MA 02139: Massachusetts Institute of Technology, Space Systems Laboratory, SSL 5-89.

Bathe, K. J. 1995. *Finite Element Procedures.* Prentice-Hall, Inc.

Crawley, E. F. and J. de Luis. 1987. "Use of Piezoelectric Actuators as Elements of Intelligent Structures", *AIAA Journal*, 25(10):1373–1385.

Crawley, E. F. and K. B. Lazarus. 1989. "Induced Strain Actuation of Isotropic and Anisotropic Plates", *AIAA Journal*, 29(6):944–951.

Ha, S. K., C. Keilers and F. K. Chang. 1992. "Finite Element Analysis of Composite Structures Containing Distributed Piezoceramics Sensors and Actuators", *AIAA Journal*, 30(3):772–780.

Hagood, N. W. and A. von Flotov. 1991. "Damping of Structural Vibrations with Piezoelectric Materials and Passive Electrical Network", *Journal of Sound and Vibration*, 146(2):243–268.

Im, S. and S. N. Atluri. 1989. "Effects of a Piezo-Actuator on a Finitely Deformed Beam Subjected to General Loading", *AIAA Journal*, 27(12): 1801–1807.

1987. *IEEE Standard on Piezoelectricity.* New York: The Institute of Electrical Electronics Engineers, Inc.

Lerch, R. 1990. "Simulation of Piezoelectric Devices by Two- and Three-Dimensional Finite Elements", *IEEE Transaction on Ultrasonics, Ferroelectrics and Frequency Control*, 37(2):233–247.

Maugin, G. A. 1988. North-Holland: "Continuum Mechanics of Electromagnetic Solids".

Mindlin, R. D. 1972a. "Elasticity, Piezoelectricity and Crystal Lattice Dynamics", *Journal of Elasticity*, 2:217–282.

Mindlin, R. D. 1972b. "High Frequency Vibrations of Piezoelectric Crystal Plates", *International Journal of Solids and Structures*, 8:895–906.

Parton, V. Z., B. A. Kudryavtsev and N. A. Senik. 1989. In "Electroelasticity" in *Applied Mechanics: Soviet Reviews. Volume 2: Electromagnetoelasticity*, G. K. Mikhailov and V. Z. Parton, eds. Hemisphere Publishing Corporation. pp. 1–58.

Tiersten, H. F. 1969. "Linear Piezoelectric Plate Vibration", Plenum Press.

Tzou, H. S. and C. I. Tseng. 1991. "Distributed Vibration Control and Identification of Coupled Elastic/Piezoelectric Systems: Finite Element Formulation and Applications", *Mechanical Systems and Signal Processing*, 5(3):215–231.

Wada, B. K., J. L. Fanson and E. F. Crawley. 1989. "Adaptive Structures", in *Adaptive Structures*, B. K. Wada, ed., New York: ASME, pp. 1–8.

Wang, B. T. and C. A. Rogers. 1991. "Modeling of Finite-Length Spatially-Distributed Induced Strain Actuators for Laminate Beams and Plates", *Journal of Intelligent Material Systems and Structures*, 2:38–58.