Analysis of Plates Subjected to In-Plane Forces Using Large Finite Elements

Analyse des plaques soumises à des forces coplanaires par la considération d'éléments de grandeur finie et de dimensions importantes

Scheibenberechnung mittels großer endlicher Elemente

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Introduction

In the analysis of plane stress problems it is very often necessary to take recourse to numerical methods. A conventional finite element analysis can give a solution to practically any plane stress problem. In such analysis, the structure is idealized as an assemblage of a large number of small elements. It can be proved that, in general, the solution obtained will approach the exact solution to the problem as the number of elements increases [1].

However, the continuum can in many problems be idealized as an assemblage of a small number of large triangles and/or rectangles. This assemblage of elements can be analysed by matrix methods provided the stiffness or flexibility matrices of the separate elements can be calculated. In this paper a derivation is presented of the stiffness matrices of large rectangular and right-angle triangular plate elements. A stiffness matrix is calculated by simple matrix manipulations with conventional finite differences (F.D.) equations.

In a conventional finite differences analysis, one F.D. equation is applied at each point of a mesh laid upon the continuum, leading to a group of simultaneous equations. Once these equations have been solved, it is an easy matter to calculate all stress components in the continuum.

Generally the following difficulties may arise in the solution of plane-stress problems by finite differences:

1. It is difficult to take into account discontinuities in the structure.
2. In a direct F.D. solution of a plane-stress problem, the structure must be

externally statically determinate. The boundary stresses are therefore known. If the structure is statically indeterminate, the direct F.D. solution must be supplemented by compatibility equations [2].

3. A conventional F.D. solution may require a large number of simultaneous equations. It may be the case that due to round-off errors or the capacity of the computer, a solution to the problem cannot be obtained.

The object of this paper is to show how the displacement method – as used in the finite element analysis – together with finite differences can overcome the difficulties mentioned above.

The plate structures considered may be of any shape, but shall have linear boundaries. They may be supported in any manner and be subjected to in-plane loading at their interior and along their boundaries. In the analysis, the structure is idealized as an assemblage of large triangular and/or rectangular elements (Fig. 1). The largest number of equations to be handled at a time are

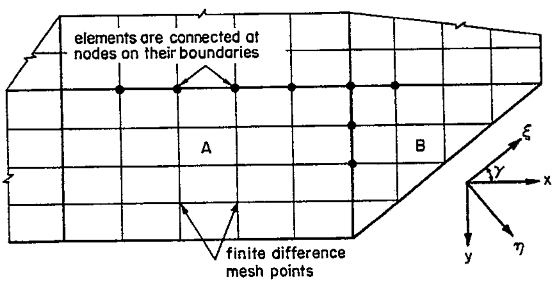


Fig. 1. Large element idealization of a plate.

either the F. D. equations used in the derivation of an element stiffness matrix or the force-displacement relations of the assembled structure, whichever is larger. Generally, the larger number of equations is required in the analysis of individual elements and this number of equations is of course, much less than what would be used in a conventional F.D. analysis of the complete structure.

The method presented here is developed to be used in the analysis of box-girder skew bridges. An analysis of such structures by conventional finite differences seems difficult if not impossible. On the other hand, a conventional solution by finite element can become expensive because of the large number of simultaneous equations involved.

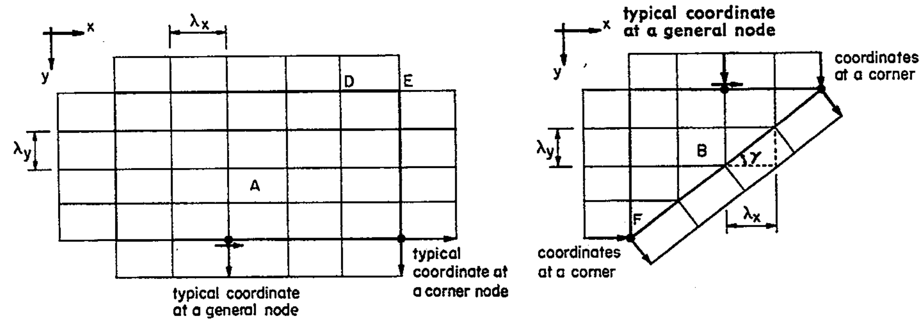
If a single-cell box-girder skew bridge is considered as an assemblage of large plate elements subjected to in-plane and bending forces, a solution by displacement method can be obtained in which finite differences are used for the derivation of the element stiffness matrices.

This paper, deals only with the plane stress problem while the stiffness matrices of plate elements corresponding to bending displacements are given elsewhere [3].

SZILARD used finite differences to derive the stiffness matrix of a square element corresponding to two in-plane coordinates at each corner [4] or at the mid points of the element edges [5]. No accurate results were obtained in two test problems solved by the use of these elements. However, SZILARD suggests that when an "optimum" number of elements is used the accuracy is improved. The elements which are used in the present paper are rectangular or triangular, the coordinates and the method of derivation of the stiffness matrix are different from SZILARD's approach. Further, the solution using these elements give accurate results.

Derivation of Element Flexibility and Stiffness Matrices

The typical rectangular and right-angle triangular elements A and B used in Fig. 1 are shown again in Fig. 2. The F.D. meshes on the elements are chosen such that mesh points on the common line between adjacent elements coincide



Number of mesh points $= 52$,
$n = 52-3 = 49$,
$m = 40$, $p = m-3 = 37$,
$r = 12$, $n = p+r$, $l = 30$,

a) Rectangular element A

Number of mesh points $= 30$,
$n = 30-3 = 27$,
$m = 27$, $p = m-3 = 24$,
$r = 3$, $n = p+r$, $l = 15$,

b) Triangular element B

Fig. 2. Examples of finite differences meshes used in the derivation of the flexibility and stiffness matrices of rectangular and triangular elements.

(Fig. 1). The adjacent elements are assumed to be connected at boundary mesh points referred to as nodes and all external forces are applied at the nodes. At a node, two degrees of freedom may be considered, representing in-plane displacements normal and tangential to the element edge (Fig. 2). At a corner point the two degrees of freedom represent displacements normal to the element edges.

The differential equation to be satisfied at all points on the element is [2]:

$$\frac{\partial^4 \Phi}{\partial x^4} + \frac{2\,\partial^4 \Phi}{\partial x^2\,\partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = 0, \tag{1}$$

where Φ is the Airy stress function. This function is related to the stresses by the equation:

$$\sigma_x = \frac{\partial^2 \Phi}{\partial y^2}, \tag{2}$$

$$\sigma_y = \frac{\partial^2 \Phi}{\partial x^2}, \tag{3}$$

$$\sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x\, \partial y}. \tag{4}$$

Using central finite differences Eq. (1) to (4) applied at any interior mesh point i can be represented in the schematic form given below, in which the horizontal and vertical lines represent F.D. mesh lines parallel to the x and y directions respectively.

$$\left[\begin{array}{ccccccccc}
 & & & & \alpha^2 & & & & \\
 & & & & | & & & & \\
 & & 2\alpha & \text{———} & -4\alpha(1+\alpha) & \text{———} & 2\alpha & & \\
 & & | & & | & & | & & \\
1 & \text{—} & -4(1+\alpha) & \text{—} & \underset{i}{6+6\alpha^2+8\alpha} & \text{—} & -4(1+\alpha) & \text{—} & 1 \\
 & & | & & | & & | & & \\
 & & 2\alpha & \text{———} & -4\alpha(1+\alpha) & \text{———} & 2\alpha & & \\
 & & & & | & & & & \\
 & & & & \alpha^2 & & & &
\end{array}\right]\{\Phi\} = 0, \tag{5}$$

$$(\sigma_x)_i = \frac{1}{\lambda_y^2}\left[\begin{array}{c} 1 \\ | \\ -2_i \\ | \\ 1 \end{array}\right]\{\Phi\}, \tag{6}$$

$$(\sigma_y)_i = \frac{1}{\lambda_x^2}[+1 \text{ — } -2_i \text{ — } +1]\{\Phi\}, \tag{7}$$

$$(\sigma_{xy})_i = \frac{1}{4\,\lambda_x \lambda_y}\left[\begin{array}{ccc} -1 & \text{———} & 1 \\ | & | & | \\ | \text{—} & i & \text{—} | \\ | & | & | \\ 1 & \text{———} & -1 \end{array}\right]\{\Phi\}, \tag{8}$$

where
$$\alpha = \left(\frac{\lambda_x}{\lambda_y}\right)^2, \tag{9}$$

λ_x and λ_y being the mesh intervals in the x and y directions (Fig. 2).

The boundary conditions at a point on a free edge parallel to the ξ-axis making an angle γ with the x-axis (Fig. 1) are:

$$(\sigma_\eta)_a = (\sigma_x)_e \sin^2\gamma + (\sigma_y)_e \cos^2\gamma + 2\,(\sigma_{xy})_e \sin\gamma\cos\gamma, \tag{10}$$

$$(\sigma_{\xi\eta})_a = (\sigma_x)_e \sin\gamma\cos\gamma - (\sigma_y)_e \sin\gamma\cos\gamma + (\sigma_{xy})_e(\cos^2\gamma - \sin^2\gamma), \tag{11}$$

in which the subscript a refers to the external applied stress and the subscript e refers to the stress as calculated by Eq. (2) to (4). In the element analysis to follow, Eq. (10) is satisfied at mesh points on the boundary while Eq. (11) is applied at mid-points between the boundary mesh points. Let the number of boundary conditions for the element be m, corresponding to m values of external applied stress σ_a.

Assume that an element is supported at any three non-parallel coordinates and a flexibility matrix corresponding to k other coordinates is required. Boundary forces $\{Q\}$ applied at the k coordinates are related to p external applied stresses $\{\sigma_a\}$ by the relations given in Fig. 3, which can be combined in one matrix equation.

$$\{\sigma_a\}_{p\times 1} = [C]_{p\times k}\{Q\}_{k\times 1}, \tag{12}$$

where

$$p = m - 3.$$

The three σ_a's omitted from the m stresses in Eq. (12) represent the equilibrants to the p-boundary applied stresses.

If a unit load is applied in turn at each of the k-coordinates the external boundary stress matrix is

$$[\sigma_a]_{p\times k} = [C]_{p\times k}. \tag{13}$$

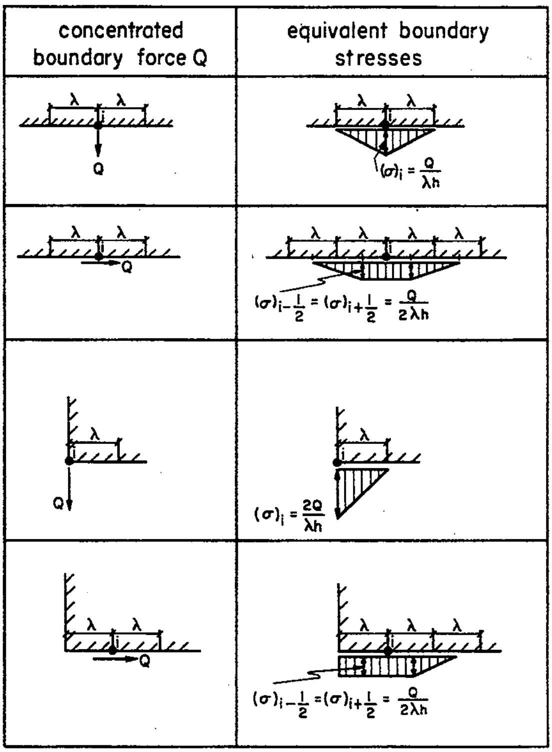


Fig. 3. Boundary stresses equivalent to concentrated forces. Thickness of plate $= h$.

Using Eq. (10) and (11) combined with Eq. (6) to (8), the p boundary stresses are related to the Airy stress function values at the mesh points by p equations. These equations together with Eq. (5) applied at the interior points lead to a group of simultaneous equations, which can be put in the form

$$[A]_{n\times n}\{\Phi\}_{n\times k} = \begin{Bmatrix} \{0\}_{r\times 1} \\ \hdashline \{\sigma_a\}_{p\times 1} \end{Bmatrix}, \tag{14}$$

where r is the number of interior mesh points and n is the total number of mesh points less three. Since the stress depends on the derivatives of ϕ, any arbitrary values for ϕ can be assigned at three points not on a straight line. Let the ϕ-values at these three points be zero. Solution of Eq. (14) gives the ϕ values at the other mesh points.

Using Eq. (6) to (8), the stresses σ_x, σ_y and σ_{xy} at the mesh points on the element can be calculated. Thus,

$$\{\sigma\}_{3l\times 1} = [E]_{3l\times n}\{\phi\}_{n\times 1}. \tag{15}$$

The element stresses corresponding to a unit load applied separately at each of the k-coordinates is given by

$$[\sigma_u]_{3l\times k} = [E]_{3l\times n}[B]_{n\times k}, \tag{16}$$

where $[B]$ is obtained by solving for $[\phi]$ in Eq. (14) with the right-hand side as follows:

$$[A]_{n\times n}[B]_{n\times k} = \begin{Bmatrix} [0]_{r\times k} \\ \hdashline [C]_{p\times k} \end{Bmatrix}. \tag{17}$$

Let the stress matrix $[\sigma_u]$ in Eq. (16) be subdivided as follows:

$$[\sigma_u]_{3l\times k} = \begin{bmatrix} [\sigma_{u1}] \\ [\sigma_{u2}] \\ \cdots \\ [\sigma_{ul}] \end{bmatrix}. \tag{18}$$

The elements of a typical submatrix $[\sigma_{ui}]_{3\times k}$ are the stresses σ_x, σ_y and σ_{xy} at mesh point i due to the k-unit load cases.

Using the "unit load" theorem [7], the flexibility matrix of the element is

$$[f]_{k\times k} = \sum_{i=1}^{l} (\Delta A)_i [\sigma_{ui}]^T [d] [\sigma_{ui}], \tag{19}$$

where

$$(\Delta A)_i = \omega_i \lambda_x \lambda_y, \tag{20}$$

and

$$[d] = \frac{h}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}. \tag{21}$$

E being Young's modulus, ν Poisson's ratio, and h is element thickness. $(\Delta A)_i$ is an elemental area around mesh point i and equals $\lambda_x \lambda_y$ for all interior points. The multiplyer $\omega_i = 1/2$, $1/4$ and $1/8$ for the typical boundary points D, E, F, respectively (Fig. 2).

Once the flexibility matrix of the element is calculated, the analysis of the assembled structure may be carried out by either the force or the displacement method. If the displacement method is chosen, the stiffness matrix of the unsupported element is to be determined from $[f]$. (See ref. [7], p. 148).

The analysis of the assembled structure gives the forces $\{Q\}$ acting at the boundary nodes of the elements. The stresses at all mesh points of an element of the assembled structure are then calculated by the equation:

$$\{\sigma\}_{3l1} = [\sigma_u]_{3lk} \{Q\}_{k1}. \tag{22}$$

General Remarks

1. While displacement continuity is preserved in the interior of elements, on the element boundaries, compatibility of displacements is achieved only at the node points. Therefore, the number of these points should be sufficiently large and the larger the elements the smaller the error caused by the idealization of the structure to an assemblage of elements. Thus, in contrast to a conventional finite element idealization, the structure should here be divided into the largest elements possible.

2. Most structures can be idealized as assemblies of rectangular and triangular elements. Sometimes, trapezoidal elements can conveniently be used, as for example in the analysis of skew bridges. The same procedure given above can be used for the trapezoidal element.

3. It is to be noted in computer programming that $[A]$, $[C]$ and $[E]$, are sparse matrices and only non zero elements should be stored or generated when needed.

4. It is possible to solve for ϕ values from a number of simultaneous equations applied at the interior points only [2]. However, for easy computer programming, the form in Eq. (14) was used.

5. Improved coefficients [8] can be used to form matrix $[A]$ in Eq. (14) instead of central finite differences. More accurate results can then be expected.

Application

Consider the continuous deep beam in Fig. 4, which has two equal spans L, unit breadth and is of depth equal to the span. For the analysis, the beam is considered as an assemblage of two equal elements connected at 7 node points along $F B$. The F.D. mesh used to derive the element stiffness matrix is of size

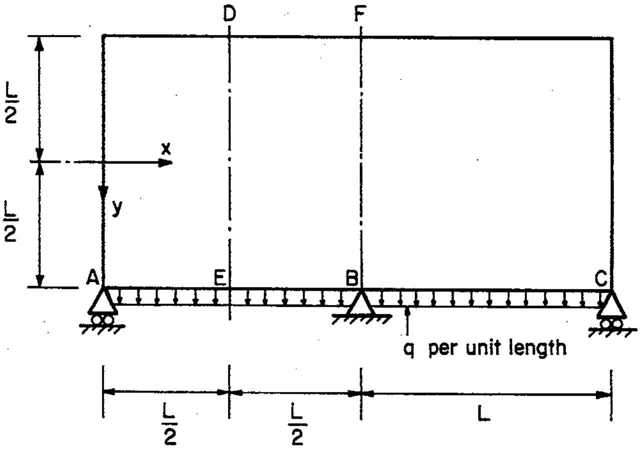
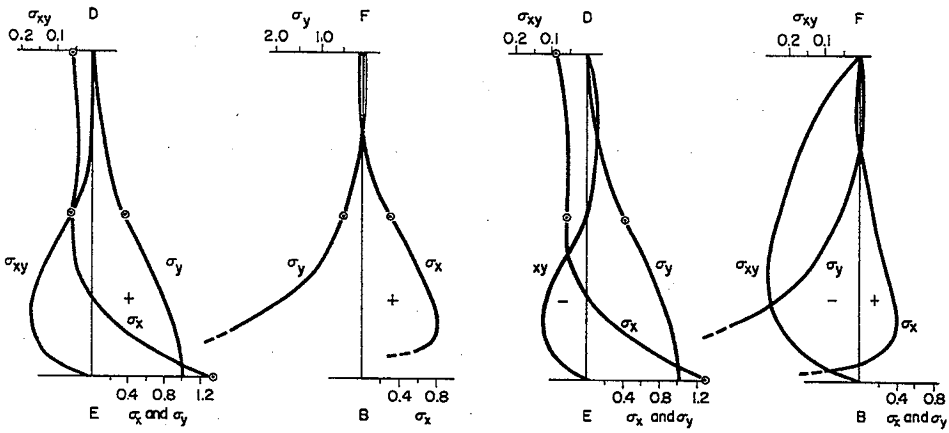


Fig. 4. Continuous plate analyzed in the example.



(a) Span AB and BC loaded (b) Span AB only loaded

⊙ Schleeh's solution (Ref. 9)

Fig. 5. Stresses in terms of q at sections DE and FB of the plate in Fig. 4.

$\lambda_x = \lambda_y = L/6$. Table 1 and Fig. 5 give the stresses calculated for a uniform load of intensity q/unit area on the bottom side of the beam in the two spans and in one span only.

The same problem was solved by SCHLEEH [9]. He determined the stresses of a statically determinate beam by a solution using Fourier series. To determine the statically indeterminate reaction at the central support, he treared a continuous beam on spring supports as an idealization of the deep beam. His extensive study shows that his method can be expected to give accurate results for continuous deep beams with depth-to-span ratio smaller or equal to unity. SCHLEEH's results (see Table 1 and Fig. 5) are in good agreement with the results of the large element analysis used in the present paper.

Table 1. Stresses in terms of q for the deep continuous beam in Fig. 4

Loading	y	Section DE			Section FB		
		σ_x	σ_y	σ_{xy}	σ_x	σ_y	σ_{xy}
Uniform load q per unit area on ABC	$-L/2$	(−0.212)* −0.245	0.000	0.000	−0.046	0.000	0.000
	$-L/3$	−0.180	0.054	0.004	−0.031	−0.055	0.000
	$-L/6$	−0.195	0.171	−0.013	0.059	−0.197	0.000
	0.0	(−0.237) −0.222	(0.367) 0.363	−0.060	(0.254) 0.250	(−0.471) −0.493	0.000
	$L/6$	−0.155	0.635	−0.135	0.600	−1.151	0.000
	$L/3$	0.236	0.915	−0.167	0.791	−2.726	0.000
	$L/2$	(1.358) 1.280	1.000	0.000	−3.291	−5.894	0.000
Uniform load q per unit area on AB	$-L/2$	(−0.364) −0.352	0.000	0.000	−0.023	0.000	0.000
	$-L/3$	−0.236	0.065	0.027	−0.016	0.038	−0.118
	$-L/6$	−0.230	0.209	0.022	0.029	−0.077	−0.167
	0.0	(−0.203) −0.241	(0.421) 0.426	−0.012	0.125	−0.237	−0.230
	$L/6$	−0.137	0.703	−0.078	0.300	−0.603	−0.254
	$L/3$	0.320	0.954	−0.126	0.396	−1.493	−0.221
	$L/2$	(1.295)* 1.399	1.000	0.000	−1.646	−3.169	0.000

* values between brackets were given by SCHLEEH [9]

Conclusion

For finite element analysis of plane-stress problems, stiffness matrices of triangular or rectangular elements are needed. If displacement functions are assumed to calculate the element stiffness matrices, it is necessary to idealize the structure as an assemblage of a large number of small elements. In this paper the stiffness matrices of the elements are derived by finite differences, without assuming displacement functions. Therefore, structures can be considered as assemblages of a small number of large elements. In fact the larger the elements used in the idealized structure, the smaller is the error caused, by element division.

The method is used in a test problem of a continuous deep beam. The results obtained show good agreement with those calculated by another method.

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Keywords

Computer; finite differences; finite elements; matrix methods; numerical methods; plane-stress problem.

Notation

$[A]$	matrix formed by finite differences coefficients of Airy stress function, Eq. (14)
$[B]$	values of Airy stress function at mesh points due to unit load applied separately at each of the boundary coordinates
$[C]$, $[E]$	transformation matrices defined by Eq. (12) and (15), respectively
$\{Q\}$	concentrated boundary forces
(ΔA)	elemental area
$[d]$	elasticity matrix (Eq. (20))
$[f]$	flexibility matrix
h	plate thickness
k	number of coordinates for which the flexibility matrix is derived
l	number of finite differences mesh points on the element
m	number of boundary stress equations for an element
n	total number of finite differences mesh points less three
p	equals $(m-3)$
r	number of interior points. Also $r=n-p$
α	constant $= (\lambda_x/\lambda_y)^2$
γ	angle between x- and ξ-axes
λ_x, λ_y	finite differences mesh spacings in the x and y directions
ν	Poisson's ratio
σ	stress; type and direction of stress defined by subscripts x, y, ξ and γ.
$\{\sigma_a\}$	applied boundary stress
$[\sigma_u]$	internal stresses at mesh points on the element due to a unit load applied separately at each of the coordinates
Φ	Airy stress function
ω	multiplying factor, Eq. (19)

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Summary

A derivation of the flexibility and the stiffness matrix of large rectangular or right-angle triangular plate elements for in-plane degrees of freedom is presented. The flexibility matrix is calculated using finite differences. This matrix is then used to derive the element stiffness matrix. The calculations involve simple matrix operations and can be easily programmed.

The stiffness matrices derived by this method are intended to be used in the analysis of structures which can be idealized as an assemblage of large rectangular or triangular plate elements, e.g. box-girder skew bridges.

Résumé

On présente une méthode permettant d'obtenir les matrices de rigidité et de flexibilité pour des éléments plans (degré de liberté = 2) en forme de rectangle ou de triangle rectangle de grandes dimensions. La matrice de flexibilité est calculée en traitant des différences finies. On déduit ensuite de cette matrice la matrice de rigidité de l'élément. Les calculs font appel à des opérations simples sur les matrices et peuvent être aisément programmés.

Les matrices de rigidité obtenues par cette méthode sont supposées être utilisées pour l'analyse de structures qui peuvent être idéalisées comme assemblage d'éléments plans de grande dimension en forme de rectangle ou de triangle rectangle, par exemple ponts inclinés à ceintre en caisson.

Zusammenfassung

Gegeben wird eine Herleitung der Flexibilitäts- und Steifigkeitsmatrix von großen rechteckigen und aus rechtwinkligen Dreiecken bestehenden Elementen für die Freiheitsgrade der Scheibe. Die Flexibilitätsmatrix ist mit Hilfe von endlichen Differenzen berechnet und zur Herleitung der Steifigkeitsmatrix verwendet worden. Die Berechnung besteht aus einfachen Matrixoperationen und kann leicht programmiert werden.

Die so hergeleitete Steifigkeitsmatrix kann für Bauwerke Anwendung finden, die durch rechteckige oder dreieckige Elemente idealisiert werden können, unter anderem für schiefe Trägerroste.