Analysis of Plates in Bending Using Large Finite Elements

Calcul des plaques fléchies à l'aide d'éléments finis de grandes dimensions

Berechnung der Biegung von Platten mittels großer endlicher Elemente

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Introduction

Solutions to plate bending problems can often only be obtained by taking recourse to numerical methods. A conventional finite element analysis can give a solution to practically any plate bending problem. In such analysis, the structure is idealized as an assemblage of a large number of small elements. It can be shown that a solution approaching the exact solution to the problem can be obtained as the number of elements used in the analysis increases [1].

In many practical plate bending problems, the continuum can be idealized to an assemblage of a small number of large rectangular and/or right angle triangular elements. For the analysis of such an assemblage, the flexibility or stiffness matrices of the large individual elements are needed. In this paper, the flexibility and the stiffness matrices of "large" rectangular and right-angle triangular elements in bending are derived by simple matrix operations with finite differences equations.

The two main difficulties arising in conventional finite differences (F.D.) analysis of complex plate bending problems may be outlined as follows:

1. To predict accurately the stress in the continuum it may be necessary to use a large number of F.D. equations. The solution of large systems of F.D. equations may become impossible due to the capacity of the computer used or due to round-off errors.

2. Boundary conditions and discontinuities of the plate can not be easily accounted for in the F.D. solution. It is mainly for this reason that in many

cases finite element analysis is preferred, although it usually involves the solution of a larger number of equations.

The object of this paper is to show how the general displacement method (as used in the finite element analysis) together with finite differences can overcome the difficulties mentioned above. The plate structures considered may be of any configuration but shall have linear boundaries. For the analysis, the structure is idealized as an assemblage of a small number of large triangular and/or rectangular elements (Fig. 1). The largest number of equations to be handled simultaneously are either the F. D. equation used in deriving the flexibility (or stiffness) of an individual element, or the force displacement relations of the assembled structure.
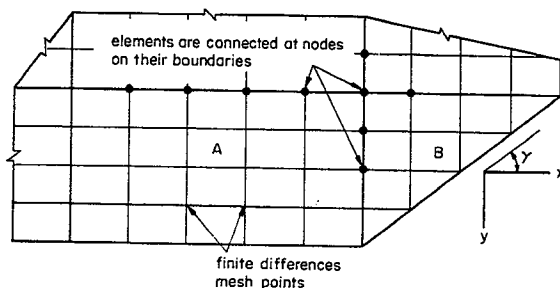

Fig. 1. Large Element Idealization of a Plate.

The method presented in this paper is developed to be used as a part of the analysis of box-girder skew bridges. An analysis of such structures by conventional finite differences appears difficult if not impossible for the reasons mentioned above. On the other hand a conventional solution by finite element may become expensive because of the large number of simultaneous equations involved.

If, for example, a single-cell box-girder skew bridge is considered as an assemblage of large plate elements subjected to in-plane and bending forces, a solution by displacement method can be obtained in which finite differences are used for the derivation of the element stiffness matrix. This paper deals only with the bending of plates while the stiffness matrix of the plate element corresponding to in-plane displacement is presented elsewhere [3].

SZILARD [4] derived the stiffness matrix of a square plate element using finite differences. He used coordinates at the centre of each element side. The structure must be divided into a large number of small elements to obtain accurate results. In the method presented in this paper, the plates are divided into elements which are essentially large (as mentioned earlier). Accuracy is increased if the finite difference mesh size within the element is reduced and larger elements are chosen. ANG and NEWMARK [2] analyzed continuous slab panels by idealizing the plate as a system of rigid bars and springs, for which

the deflection and load are related by equations identical to the F.D. equations of the plate. First, the deflection of each slab panel is calculated assuming the edges to be fixed. Then the panel boundary displacements could be determined using a relaxation technique. In this manner, the solution of a large number of F.D. equations was avoided.

In the following section, the flexibility and stiffness matrices of rectangular and triangular elements are derived.

Derivation of Element Flexibility and Stiffness Matrices

The large elements idealizing the plate are assumed to be connected at a finite number of node points along their boundaries (Fig. 1). Fig. 2 shows a typical rectangular element A and a right-angle triangular element B. A F.D. mesh is chosen such that node points on the lines between the elements in the assembled plate coincide with F.D. mesh points. At node points on the element edges two degrees of freedom may be considered, a transverse deflection and
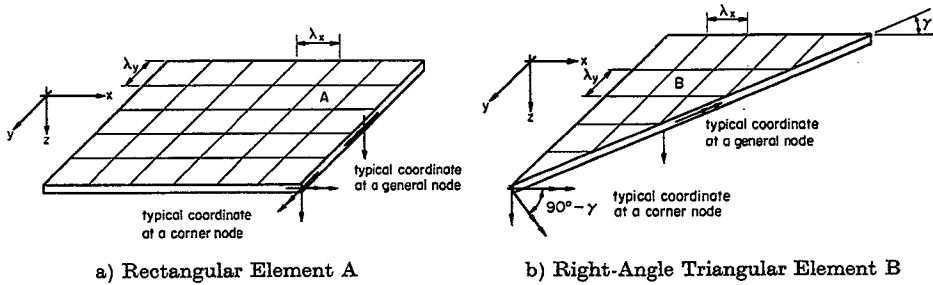

a) Rectangular Element A   b) Right-Angle Triangular Element B

Fig. 2. Typical Coordinates at Nodes on Edges ant at Corners.

a rotation about an axis along the edge. At a corner node point, the degrees of freedom can be a transverse deflection and rotations about axes perpendicular to the two edges meeting at that corner. Compatibility of displacement of elements will be achieved at these nodes at the chosen coordinates (Fig. 1). If the boundary nodes are sufficiently close, the deformation of the assembled elements will represent that of the actual slab.

The differential equation to be satisfied at any mesh point on the element is [5]

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{N}, \qquad (1)$$

where w is the deflection of the plate; q intensity of a distributed transverse applied load;

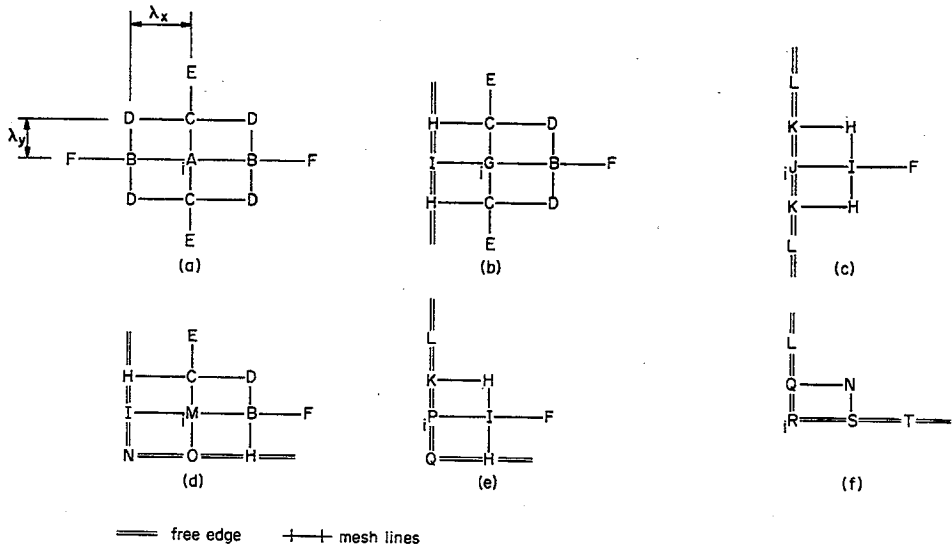$$N = \frac{E h^3}{12(1-\nu^2)}. \qquad (2)$$

E is the modulus of elasticity; h the thickness (assumed constant for each element) and ν Poisson's ratio. Using central finite differences, Eq. (1) applied at a typical mesh point i, away from the boundary, can be written in the form [6]:

$$N \frac{\lambda_y}{\lambda_x^3} [A] \{w\} = Q_i, \qquad (3)$$

where $[A]$ is a row matrix of dimensionless coefficients of the deflection at i and at 12 other points in its vicinity;

$$Q_i = q_i \lambda_x \lambda_y, \qquad (4)$$

λ_x and λ_y being the mesh spacings in the x and y directions (Fig. 2).

Fig. 3. Finite Difference Coefficients for a Rectangular Plate with Free Edges. Coefficients to be Used in the Equation:

$$\left(\frac{N\lambda_y}{\lambda_x^3}\right) [\text{coefficients}] \{w\} = Q_i.$$

$A = 6 + 6\alpha^2 + 8\alpha$

$B = -4(1+\alpha)$

$C = -4\alpha(1+\alpha)$

$D = 2\alpha$

$E = \alpha^2$

$F = 1$

$G = 5 + 6\alpha^2 + 8\alpha$

$H = \alpha(2-\nu)$

$I = -2(2\alpha - \nu\alpha + 1)$

$J = 1 + 4\alpha(1-\nu) + 3\alpha^2(1-\nu^2)$

$K = -2\alpha[1 - \nu + \alpha(1-\nu^2)]$

$L = \frac{1}{2}\alpha^2(1-\nu^2)$

$M = 5 + 5\alpha^2 + 8\alpha$

$N = 2\alpha(1-\nu)$

$O = -2\alpha(2 - \nu + \alpha)$

$P = 1 + 4\alpha(1-\nu) + \frac{5}{2}\alpha^2(1-\nu^2)$

$Q = -2\alpha\left[1 - \nu + \frac{\alpha}{2}(1-\nu^2)\right]$

$R = 2\alpha(1-\nu) + \frac{1}{2}(1+\alpha^2)(1-\nu^2)$

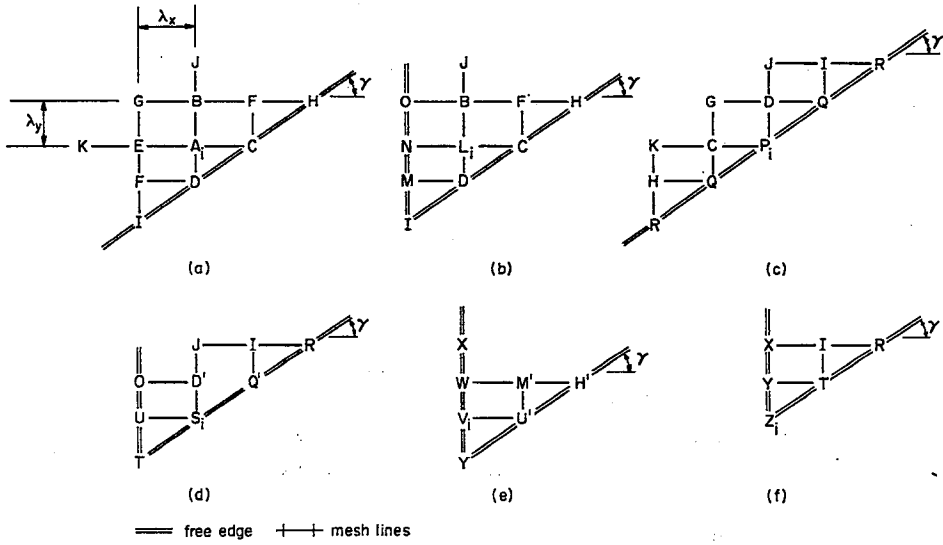$S = -2\left[\alpha(1-\nu) + \frac{1}{2}(1-\nu^2)\right]$

$T = \frac{1}{2}(1-\nu^2)$

Where $\alpha = \left(\frac{\lambda_x}{\lambda_y}\right)^2$

=== free edge   +—+ mesh lines

$A = 5 + 5\alpha^2 + 8\alpha$

$B = -4\alpha(1+\alpha)$

$C = -2(1+\alpha) - \epsilon(2-\alpha)$

$D = -2\alpha(1+\alpha) - \epsilon(2\alpha-1)$

$D' = -2\alpha(1+\alpha) - \epsilon(2\alpha-1) + \epsilon c$

$E = -4(1+\alpha)$

$F = \alpha$

$G = 2\alpha$

$H = \epsilon$

$H' = \epsilon - c\nu(1-\nu)$

$I = \epsilon\alpha$

$J = \alpha^2$

$K = 1$

$L = 4 + 5\alpha^2 + 8\alpha$

$M = \alpha(1-\nu)$

$M' = \alpha(1-\nu) - c\alpha(1-\nu)$

$N = -2(2\alpha - \nu\alpha + 1)$

$O = \alpha(2-\nu)$

$P = 1 + \alpha^2 + 4\epsilon(1+\alpha) - 3\epsilon^2$

$Q = 2\epsilon^2 + \alpha - 2\epsilon(1+\alpha)$

$Q' = 2\epsilon^2 + \alpha - 2\epsilon(1+\alpha) + \dfrac{\epsilon c \nu}{\alpha}$

$R = -\dfrac{1}{2}\epsilon^2$

$S = \alpha^2 + 4\epsilon(1+\alpha) - \dfrac{5}{2}\epsilon^2 - 2\epsilon n\left(1+\dfrac{\nu}{\alpha}\right) + \dfrac{\epsilon\nu}{\alpha}(1-c)$

$T = \alpha(1-\nu-2\epsilon) + \epsilon^2 - 2\epsilon(1-n)\left(1+\dfrac{\nu}{\alpha}\right) + \dfrac{\nu\epsilon}{\alpha}$

$U = -2\alpha(1-\nu) - 2\epsilon + \epsilon\alpha + \epsilon(1-c)$

$U' = -2\alpha(1-\nu) - 2\epsilon + \alpha(1-\nu)\left[2n\left(1+\dfrac{\nu}{\alpha}\right) - \dfrac{\nu}{\alpha}(1-c)\right]$

$V = \dfrac{5}{2}\alpha^2 + 4\alpha(1-\nu) - \dfrac{5}{2}\nu^2\alpha^2 - \alpha(1-c)(1-\nu)$

$W = -2\alpha[1-\nu+\alpha(1-\nu^2)]$

$X = \dfrac{1}{2}\alpha^2(1-\nu^2)$

$Y = -2\alpha[1-\nu+\alpha(1-\nu^2)] + \epsilon + \alpha^2(1-\nu^2)$

$\qquad + \alpha(1-\nu)\left[2(1-n)\left(1+\dfrac{\nu}{\alpha}\right) - \dfrac{\nu}{\alpha}\right]$

$Z = -(X+I+R+Y+T)$

Where $\alpha = \left(\dfrac{\lambda_x}{\lambda_y}\right)^2$, $n = \cos^2\gamma$, $\epsilon = n(1-\nu)$, $c = 1-2\sin^2\gamma$

Fig. 4. Deflection Coefficients of the F.D. Equation Applied at Point i on or Adjacent to a Free Edge of a Triangular Element. F.D. Equation Takes the Form:

$$\left(\dfrac{N\lambda_y}{\lambda_x^3}\right) [\text{coefficients}] \{w\} = Q_i.$$

The pattern of the deflection coefficients of Eq. (3) (elements of $[A]$) are shown in Fig. 3a. When the point i where the F.D. equation is applied is at or adjacent to the boundary, the coefficients must be modified to incorporate the boundary condition of a free edge [6]. The modified coefficients are given in Fig. 3b to f. The corresponding coefficients for the triangular element (Fig. 2b) are given in Fig. 4.

The F.D. equations applied at all mesh points of an element with free edges may be written in matrix form

$$[K]\{w\} = \{Q\}, \tag{5}$$

where $[K]$ is a square symmetric matrix formed by F.D. coefficients. It relates the deflection and the forces applied at the mesh points and can therefore be regarded as a stiffness matrix corresponding to the coordinates $\{w\}$. The matrix $[K]$ is singular. The element must be restrained at not less than three coordinates (not on a straight line) in order that its stiffness matrix can be inverted. If springs of arbitrary stiffness (say $N\lambda_y/\lambda_x^3$), are introduced at three (or more) boundary nodes, then the spring stiffness is added to the corresponding elements in $[K]$ to obtain $[K^*]$ which can be inverted. Thus,

$$\{w\} = [K^*]^{-1}\{Q\}. \tag{6}$$

Flexibility matrix $[K^*]^{-1}$ will now be transformed into another flexibility matrix $[f^*]$ corresponding to the degrees of freedom at the boundary (Fig. 2). The displacements $\{D\}$ at the coordinates in Fig. 2 are related by geometry to the deflections $\{w\}$ at the element mesh points as follows:

$$\{D\} = [C]\{w\}, \tag{7}$$

Therefore, the flexibility matrix of the element on spring supports corresponding to the $\{D\}$-coordinates is given by [7]
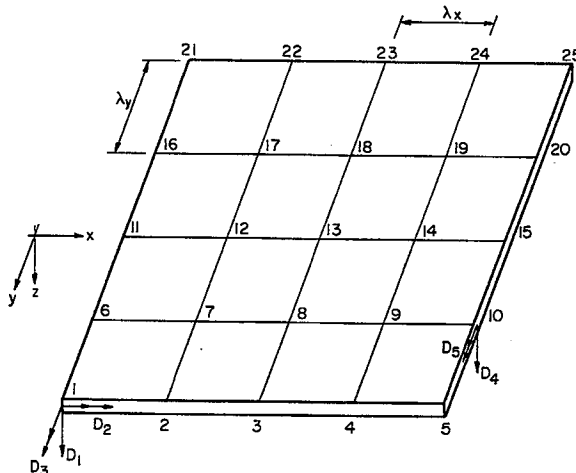

Fig. 5. Mesh Points and Coordinates Referred to in Equation 7a.

$$[f^*] = [C][K^*]^{-1}[C]^T, \tag{8}$$

where the superscript T means matrix transpose.

The elements of $[C]$ are either zero, 1, $1/\lambda_x$, or $1/\lambda_y$. Consider for example the plate in Fig. 5 which has 25 mesh points. The edge displacements $\{D\}$ at the five arbitrary coordinates shown in the figure are related to the deflections $\{w\}$ at the 25 mesh points by Eq. (7) as follows:

Column

$$\text{Number} \quad 1 \qquad 2 \quad \ldots \quad 6 \quad \ldots \quad 9 \qquad 10 \quad \ldots \quad 25$$

$$\{D\}_{5 \times 1} = \begin{bmatrix} 1 & & & & & & \\ \dfrac{1}{\lambda_y} & & & -\dfrac{1}{\lambda_y} & & \text{Elements not} & \\ \dfrac{1}{\lambda_x} & -\dfrac{1}{\lambda_x} & & & & \text{shown are zero} & \\ & & & & & 1 & \\ & & & & \dfrac{1}{\lambda_x} & -\dfrac{1}{\lambda_x} & \end{bmatrix} \{w\}_{25 \times 1} \tag{9}$$

or
$$\{D\} = [C]\{w\}. \tag{9a}$$

In this equation any of the edge rotations, say D_5 (Fig. 5) is considered equal to the slope of the deflected surface midway between mesh points 10 and 9. This would be accurate if the mesh size λ_x is small or the curvature perpendicular to the free edge is small. Using $[C]$ in the above form in Eq. (9) would give a less accurate value for the elements f_{22}^*, f_{33}^* and f_{55}^* than the other elements of the matrix $[f^*]$. This is because these elements represent the rotation at a coordinate due to a unit couple applied at the same coordinate. A more accurate value of any of these elements, say f_{55}^*, is given by:

Colomn

$$\text{Number} \quad 1 \quad \ldots \quad 7 \quad 8 \quad 9 \quad 10 \quad \ldots \quad 25$$

$$f_{55}^* = \left(\frac{1}{6\lambda_x}\right) \begin{bmatrix} 0 & 0 & 2 & -9 & 18 & -11 & 0 & \ldots & 0 \end{bmatrix} [K^*]^{-1} \begin{Bmatrix} 0 \\ \ldots \\ \ldots \\ 0 \\ \dfrac{1}{\lambda_x} \\ -\dfrac{1}{\lambda_x} \\ 0 \\ \ldots \\ 0 \end{Bmatrix} \begin{matrix} \text{Row} \\ \text{Number} \\ 1 \\ \\ \\ \\ 9 \\ \\ 10 \\ \\ \\ 25 \end{matrix} \tag{10}$$

In this equation the slope at the element edge is calculated from the deflec-
tions using the pattern of coefficients [8] $(1/6\lambda_x)$ [11, −18, 9, −2] and a unit
couple at the edge is replaced by a pair of equal and opposite forces (of mag-
nitude $1/\lambda_x$) at mesh points 9 and 10.

The flexibility matrix $[f^*]$ derived above (Eq. (8)) is for an element on
spring supports.

From $[f^*]$ the flexibility matrix of the element supported in any manner
can be derived [7]. The inversion of $[f^*]$ gives the stiffness matrix $[S^*]$ of the
element on spring supports. When the arbitrarily chosen stiffness of the
springs are deducted from the appropriate diagonal elements of the latter
matrix, the stiffness matrix $[S]$ of the free (unsupported) element is obtained.

Fixed-end Forces

If the displacement method of analysis is used, the fixed-end forces $\{F_r\}$
at the $\{D\}$ coordinates will be needed for any transverse loading $\{Q\}$ on the
element.

Combining Eq. (6) and (7), the node displacements of the element on spring
supports due to forces at mesh points can be written as

$$\{D\} = [C][K^*]^{-1}\{Q\}. \tag{11}$$

These node displacements are reduced to zero by the restraining forces
$\{F_r\}$. Hence,

$$[f^*]\{F_r\} + [C][K^*]^{-1}\{Q\} = \{0\},$$

from which

$$\{F_r\} = -[f^*]^{-1}[C][K^*]^{-1}\{Q\}. \tag{12}$$

Stress Resultants

The stress resultants at any mesh point can be calculated by F.D. Eqs. [7]
from the final deflections $\{w\}$. In the displacement method, the final deflections
are given by the superposition equation:

$$\{w\} = \{w_r\} + [w_u]\{D\}, \tag{13}$$

where $\{w_r\}$ are mesh point deflections of the element with restraint edges and
$[w_u]$ are mesh point deflections corresponding to unit nodal displacement, and
$\{D\}$ are the final nodal displacements. Eq. (13) can be put in the form:

$$\{w\} = [K^*]^{-1}(\{Q\} + [C]^T\{F_r + [f^*]^{-1}\{D\}\}). \tag{14}$$

General Remarks

1. A study of the finite difference coefficients in Fig. 4 shows that the stiffness matrix $[K]$ for a triangular element is not quite symmetric. The coefficients introducing unsymmetry in the matrix have been marked with a prime. For example, for symmetry, D' in Fig. 4d should be equal to D in Fig. 4b. The difficulty of obtaining finite differences coefficients which have reciprocal relationships at mesh points near skew edge corners is reported by JENSEN [6]. To overcome this difficulty, it is suggested that unsymmetric elements of $[K]$ are to be replaced by their average.

2. It should be noted that $[f^*]$ can only be inverted if a displacement at each of the D-coordinates can be imposed while the displacement at the other D-coordinates is prevented. For example, the rotation D_3 in Fig. 5, which is expressed in terms of the transverse deflection at points in edge 1–5, cannot be produced if at the same time the transverse deflection at points 1, 2, 3 and 4 are zero. This does not provide serious difficulty. It only has to be noted when choosing the system of the D-coordinates representing the element degrees of freedom.

3. The finite differences patterns of coefficients required for the derivation of the flexibility or stiffness matrices of rectangular and triangular elements are included in Fig. 3 and 4. However, in some cases, some saving in the computations may be achieved by using trapezoidal elements. The writers used trapezoidal elements for the analysis of skew bridges.

Application

To test the above method, a uniformly-loaded rectangular plate with two simply-supported edges, a built-in edge and a free edge was analyzed (Fig. 6). The plate was idealized as an assemblage of two elements A and B, with the F.D. mesh shown in Fig. 6. The finer mesh in the y-direction for element B

Table 1. Moments and Deflections of Plate in Fig. 6 ($\nu = 0.3$)

Mesh	Section CD		Section EF		
	Point	Moment M_y in terms of qL^2	Point	Moment M_x in terms of qL^2	Deflections in terms of $\frac{qL^4}{Eh^3}$
4×4	2 4	−0.097 −0.134	6 8	0.091 0.121	0.114 0.159
6×6	4	−0.132	8	0.122	0.157
8×8	4	−0.130	8	0.123	0.156
Exact	4	−0.124	8	0.123	0.154

Fig. 6. Plate Used for Test Problem.

was chosen to get accurate values of M_y, which is known to change rapidly near the fixed edge. The two elements are connected at two coordinates at each of the internal mesh points on line GH.

Table 1 gives moments and deflections along CD and EF obtained by a solution in which a 4×4 mesh was taken for each of elements A and B. The table also includes the results of two other solutions in which the meshes were 6×6 and 8×8 instead of 4×4.

Conclusion

In conventional finite element analysis of a plate in bending, displacement functions are assumed to calculate the element stiffness matrices. For accurate results, it is necessary to idealize the plate as an assemblage of a (comparatively) large number of small elements. In this paper, the stiffness matrix is derived by finite differences, without assuming a displacement function. Therefore, the plate can be considered as an assemblage of a small number of large elements. In fact, the larger the elements used in the idealized structure, the smaller is the error caused by element division.

The results of the test problem show that the method gives accurate answers.

Acknowledgment

Keywords

Computer matrix methods; finite differences; finite elements; numerical methods; plate bending; slabs.

Notation

$[C]$ transformation matrix defined in Eq. (7)
$\{D\}$ displacements at nodes on element edges
$[f]$ flexibility matrix corresponding to the $\{D\}$-coordinates
$\{F_r\}$ forces at the $\{D\}$-coordinates when $\{D\}=\{0\}$
$[K]$ "equivalent" stiffness matrix formed by F.D. coefficients (Eq. (5))
N flexural rigidity of plate (Eq. (2))
$\{Q\}$ concentrated mesh point transverse loads
q intensity of distributed load
$[S]$ stiffness matrix corresponding to the $\{D\}$-coordinates
$\{w\}$ mesh points transverse deflections
λ spacing between F.D. mesh lines
ν Poisson's ratio

Subscripts and Superscripts:

$*$ used as superscripts in $[K^*]$, $[f^*]$ and $[S^*]$ to refer to plate element on arbitrary chosen spring supports
r refers to restrained element, that is – displacements $\{D\}=\{0\}$
x, y rectangular coordinate axes

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Summary

Plates in bending are analyzed by idealizing the continuum as an assemblage of large rectangular and triangular elements. The flexibility and stiffness matrices of these elements corresponding to boundary displacements are derived using finite differences. The analysis involves simple matrix operations which can be easily computer programmed.

The stiffness matrices derived by this method are intended for use in the analysis of structures which can be idealized as an assemblage of large rectangular and triangular plate elements, e. g. box-girder skew bridges.

Résumé

· On calcule les plaques fléchies au moyen d'une idéalisation du continu par un assemblage d'éléments rectangulaires et triangulaires de grandes dimensions. Les matrices de flexibilité et de rigidité de ces éléments pour les déplacements au contour sont obtenues à l'aide du calcul aux différences finies. Le calcul se fait à l'aide d'opérations matricielles simples et peut aisément être programmé.

Les matrices de rigidité assemblées par cette méthode sont destinées au calcul de structures qu'on peut idéaliser au moyen de grands éléments rectangulaires ou triangulaires, p. ex. des ponts biais à section fermée.

Zusammenfassung

Die Biegung der Platten wird durch eine Diskretisation des Kontinuums in eine Reihe von rechteckigen und dreieckigen Elementen großer Abmessungen berechnet. Die für die Randverschiebungen ausgedrückten Flexibilitäts- und Steifigkeitsmatrizen werden anhand endlicher Differenzen ermittelt. Die Analyse umfaßt einfache Matrix-Operationen, die für den Computer leicht programmierbar sind.

Die durch diese Methode abgeleiteten Streifigkeitsmatrizen sind für Tragwerke bestimmt, die zusammengesetzt aus großen rechteckigen und dreieckigen Plattenelementen idealisiert werden können, z. B. Kastenträger von schiefen Brücken.