EARTHQUAKE ENGINEERING AND STRUCTURAL DYNAMICS, VOL. 1, 241–252 (1973)

NONLINEAR DYNAMIC ANALYSIS OF COMPLEX STRUCTURES

E. L. WILSON†
University of California, Berkeley, California

I. FARHOOMAND‡
John Blume and Associates, San Francisco, California

K. J. BATHE§
University of California, Berkeley, California

SUMMARY

A general step-by-step solution technique is presented for the evaluation of the dynamic response of structural systems with physical and geometrical nonlinearities. The algorithm is stable for all time increments and in the analysis of linear systems introduces a predictable amount of error for a specified time step. Guidelines are given for the selection of the time step size for different types of dynamic loadings. The method can be applied to the static and dynamic analysis of both discrete structural systems and continuous solids idealized as an assemblage of finite elements. Results of several nonlinear analyses are presented and compared with results obtained by other methods and from experiments.

INTRODUCTION

The development of numerical methods for the nonlinear analysis of structures has attracted much attention during the past several years.[1–6] Most of the investigations have been concerned with the analysis of a particular type of structure and nonlinearity. The purpose of this paper is to present a general solution scheme for the static and dynamic analysis of an arbitrary assemblage of structural elements with both physical and geometrical nonlinearities. The structural elements may be beam elements or two- and three-dimensional finite elements which are used to idealize continuous solids.

There are various approximations involved in the representation of a continuous body as an assemblage of finite elements.[7] In this paper only the errors associated with the solution of the discrete system nonlinear equations of equilibrium are discussed. An incremental form of the equations is given, which can be used to obtain a check of equilibrium in the deformed configuration.

For dynamic analysis an efficient algorithm for the integration of the equations of motion is needed. Various different techniques are in use.[8,9] As is well known, the cost of an analysis relates directly to the size of the time step which has to be used for stability and accuracy. In this paper an unconditionally stable scheme is presented, which therefore can allow a relatively large time step to be used. Naturally, the accuracy of the numerical integration depends on the size of the time step. For linear systems, the errors associated with the numerical integration result in elongation of the free vibration periods and in decrease of the vibration amplitudes. With this in mind, guidelines can be given for the selection of an appropriate time step size in a practical analysis.

INCREMENTAL FORM OF EQUATIONS OF MOTION

The dynamic force equilibrium at the nodes (or joints) of a system of structural elements at any time can be written as

$$\mathbf{F}^i+\mathbf{F}^d+\mathbf{F}^e = \mathbf{R} \tag{1}$$

† Associate Professor of Civil Engineering.
‡ Senior Research Engineer.
§ Assistant Research Engineer.

Received 28 March 1972

where

$\mathbf{F}^i$ = inertia force vector
$\mathbf{F}^d$ = damping force vector
$\mathbf{F}^e$ = internal resisting force vector
$\mathbf{R}$ = vector of externally applied forces

Linear systems

For linear systems the force vectors can be expressed directly in terms of the physical properties of the structural elements, namely

$$\mathbf{F}^i = \mathbf{M}\ddot{\mathbf{u}}, \quad \mathbf{F}^d = \mathbf{C}\dot{\mathbf{u}}, \quad \mathbf{F}^e = \mathbf{K}\mathbf{u} \tag{2}$$

where **M, C** and **K** are the mass, stiffness and damping matrices, and $\mathbf{u}$, $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$ are the nodal point displacement, velocity and acceleration vectors of the system. The elements in **M**, **C** and **K** are constant, so that equation (1) which may be rewritten as

$$\mathbf{M}\ddot{\mathbf{u}}+\mathbf{C}\dot{\mathbf{u}}+\mathbf{K}\mathbf{u} = \mathbf{R} \tag{3}$$

constitutes a set of linear differential equations in the displacement vector **u**.

If the damping of the system is assumed to be of a restricted form which does not introduce modal coupling,[10] equation (3) can be solved by the mode superposition method. However, the step-by-step integration presented later may give a more efficient solution in cases where a large number of modes participate in the response.

Nonlinear systems

In the case of nonlinear behaviour, equation (1) is conveniently written at time $t+\Delta t$ as

$$(\mathbf{F}_t^i+\Delta\mathbf{F}_t^i)+(\mathbf{F}_t^d+\Delta\mathbf{F}_t^d)+(\mathbf{F}_t^e+\Delta\mathbf{F}_t^e) = \mathbf{R}_{t+\Delta t} \tag{4}$$

where the subscript t denotes the time at the beginning of the time increment Δt. The force vectors $\mathbf{F}_t^i$, $\mathbf{F}_t^d$ and $\mathbf{F}_t^e$ need to be evaluated using the displacements, velocities and accelerations at time t. The force changes over the time interval Δt are assumed to be given by

$$\Delta\mathbf{F}_t^i = \mathbf{M}_t\Delta\ddot{\mathbf{u}}_t, \quad \Delta\mathbf{F}_t^d = \mathbf{C}_t\Delta\dot{\mathbf{u}}_t, \quad \Delta\mathbf{F}_t^e = \mathbf{K}_t\Delta\mathbf{u}_t \tag{5}$$

where $\mathbf{M}_t$, $\mathbf{C}_t$ and $\mathbf{K}_t$ are the mass, damping and stiffness matrices at time t; $\Delta\ddot{\mathbf{u}}$, $\Delta\dot{\mathbf{u}}$ and $\Delta\mathbf{u}$ are the changes in the accelerations, velocities and displacements during the time increment. Hence equation (4) becomes

$$\mathbf{M}_t\Delta\ddot{\mathbf{u}}_t+\mathbf{C}_t\Delta\dot{\mathbf{u}}_t+\mathbf{K}_t\Delta\mathbf{u}_t = \mathbf{R}_{t+\Delta t}^* \tag{6}$$

where

$$\mathbf{R}_{t+\Delta t}^* = \mathbf{R}_{t+\Delta t}-\mathbf{F}_t^i-\mathbf{F}_t^d-\mathbf{F}_t^e \tag{7}$$

The numerical integration scheme to be presented relates $\Delta\ddot{\mathbf{u}}_t$ and $\Delta\dot{\mathbf{u}}_t$ to $\Delta\mathbf{u}_t$. Therefore equation (6) can be solved for $\Delta\mathbf{u}_t$. This also gives $\Delta\dot{\mathbf{u}}_t$ and $\Delta\ddot{\mathbf{u}}_t$.

It should be noted that the relations in equation (5) are only approximations. But the residual force $\mathbf{R}_{t+\Delta t}^r$, given by

$$\mathbf{R}_{t+\Delta t}^r = \mathbf{R}_{t+\Delta t}-\mathbf{F}_{t+\Delta t}^i-\mathbf{F}_{t+\Delta t}^d-\mathbf{F}_{t+\Delta t}^e \tag{8}$$

is a measure of how well equilibrium is satisfied at time $t+\Delta t$. In order to satisfy equilibrium to a prescribed tolerance at the end of each time step, it may be necessary to use iteration.

EVALUATION OF MATRICES FOR NONLINEAR SYSTEMS

In the preceding section nonlinear mass, damping and stiffness effects have been considered. The solution procedure is now specialized to the analysis of systems with nonlinear stiffness only. This is the most frequent requirement. In this case equation (6) becomes

$$\mathbf{M}\Delta\ddot{\mathbf{u}}_t+\mathbf{C}\Delta\dot{\mathbf{u}}_t+\mathbf{K}_t\Delta\mathbf{u}_t = \mathbf{R}_{t+\Delta t}-\mathbf{M}\ddot{\mathbf{u}}_t-\mathbf{C}\dot{\mathbf{u}}_t-\mathbf{F}_t^e \tag{9}$$

where **M** and **C** are the constant mass and damping matrices used in equation (2). The evaluation of matrices $\mathbf{K}_t$ and $\mathbf{F}_t^e$ is discussed below.

The tangent stiffness matrix $\mathbf{K}_t$

The tangent stiffness matrix of an element at a particular time is the sum of the incremental stiffness matrix $\mathbf{K}_i$ and the geometric stiffness matrix $\mathbf{K}_g$

$$\mathbf{K}_t = \mathbf{K}_i + \mathbf{K}_g \tag{10}$$

The calculation of $\mathbf{K}_i$ for each element follows the standard approach.[6] Note that the calculation must be performed in the deformed geometry. Also, for nonlinear materials and large strains the incremental stress–strain relationship associated with the strains at that time must be used.

Many investigators have derived geometric stiffness matrices for various structural components.[4–6] The general mathematical expression of virtual work which leads to the definition of the matrix is

$$\delta W = \int_{Vol} \tau_{ij}\, \delta\eta_{ij}\, \mathrm{d}V \tag{11}$$

in which τ_{ij} is the stress in the deformed position and η_{ij} is the quadratic part of the strain tensor. It follows that the geometric stiffness matrix $\mathbf{K}_g$ can be negligible as compared to $\mathbf{K}_i$ when the magnitude of the stresses is not large. However, at a high stress level this stiffness effect can be significant, and the distribution of the stresses should be defined accurately.

The suggested approach to evaluate the tangent stiffness matrix $\mathbf{K}_t$ of an element at a given time is as follows:

1. Compute the nodal point displacements and the co-ordinates of the nodal points of the element in the deformed position.
2. Compute total large strains using the 'exact' nonlinear strain-displacement relationship.
3. From the appropriate stress–strain relation and the history of strain calculate the new material constants and the stresses which correspond to that strain level and strain rate.
4. Compute the incremental stiffness matrix based on the incremental properties at that state of stress.
5. Compute the geometric stiffness matrix and add to the incremental stiffness matrix to obtain the tangent stiffness matrix of the element.

The internal resisting force vector $\mathbf{F}_t^e$

It is possible to calculate the force vector $\mathbf{F}_t^e$ by simply adding up the incremental force changes $\Delta\mathbf{F}_t^e$. However, because the stiffness matrix $\mathbf{K}_t$ is, in general, only an approximation, significant errors can accumulate in this procedure.

It is preferable to compute the force vector $\mathbf{F}_t^e$ directly using the virtual work principle.[11] This principle leads for the discrete element system at time t to the following equation

$$\delta\mathbf{u}^{\mathrm{T}}\mathbf{F}_t^e = \sum \int_{Vol} \tau_{ij}\, \delta\varepsilon_{ij}\, \mathrm{d}v \tag{12}$$

where $\delta\mathbf{u}$ is a virtual nodal displacement vector, $\delta\varepsilon_{ij}$ is the corresponding virtual small strain and τ_{ij} is the actual element stress in the deformed configuration (force per unit of deformed area). The summation sign indicates that the integral is computed over the volume of all elements.

Therefore, the vector $\mathbf{F}_t^e$ is obtained as follows:

1. Within each element calculate the strain distribution from the total displacements by the direct application of the 'exact' nonlinear strain-displacement equations. If the change in geometry of the structural system is not appreciable, only the linear part of the strain-displacement equations need be considered.
2. From the appropriate large deformation stress–strain relationships compute the corresponding stresses. These stresses exist in the deformed geometry of the structural system. For systems with material nonlinearities the stresses may be history dependent.
3. Using equation (12) the nodal forces of each element can be calculated from the stress distribution obtained in step 2.

Similar to the calculation of the system tangent stiffness matrix from the element matrices, the vector $\mathbf{F}_t^e$ is now formed by a direct assembly of the element force vectors.

SOLUTION OF EQUATIONS

In this section the step-by-step integration method for solving the equations of motion is presented. The accuracy which can be obtained in the numerical integration is studied, and guidelines are given for the selection of the size of the time step.

The step-by-step integration

Let $\mathbf{u}_t$, $\dot{\mathbf{u}}_t$ and $\ddot{\mathbf{u}}_t$ be known vectors. To obtain the solution at time $t+\Delta t$, we assume that the acceleration varies linearly over the time interval $\tau = \theta\Delta t$, where $\theta \geqslant 1{\cdot}0$. When $\theta = 1{\cdot}0$ the algorithm reduces to the standard linear acceleration method. However, as discussed later, a more suitable θ should be used.

Using the linear acceleration assumption it follows that

$$\dot{\mathbf{u}}_{t+\tau} = \dot{\mathbf{u}}_t + \frac{\tau}{2}(\ddot{\mathbf{u}}_{t+\tau} + \ddot{\mathbf{u}}_t) \tag{13}$$

$$\mathbf{u}_{t+\tau} = \mathbf{u}_t + \tau\dot{\mathbf{u}}_t + \frac{\tau^2}{6}(\ddot{\mathbf{u}}_{t+\tau} + 2\ddot{\mathbf{u}}_t) \tag{14}$$

which gives

$$\ddot{\mathbf{u}}_{t+\tau} = \frac{6}{\tau^2}(\mathbf{u}_{t+\tau} - \mathbf{u}_t) - \frac{6}{\tau}\dot{\mathbf{u}}_t - 2\ddot{\mathbf{u}}_t \tag{15}$$

and

$$\dot{\mathbf{u}}_{t+\tau} = \frac{3}{\tau}(\mathbf{u}_{t+\tau} - \mathbf{u}_t) - 2\dot{\mathbf{u}}_t - \frac{\tau}{2}\ddot{\mathbf{u}}_t \tag{16}$$

The equations of motion, equations (3) and (9), shall be satisfied at time $t+\tau$; therefore we have

$$\mathbf{M}\ddot{\mathbf{u}}_{t+\tau} + \mathbf{C}\dot{\mathbf{u}}_{t+\tau} + \mathbf{K}\mathbf{u}_{t+\tau} = \tilde{\mathbf{R}}_{t+\tau} \tag{3*}$$

and

$$\mathbf{M}\Delta\ddot{\mathbf{u}}_t + \mathbf{C}\Delta\dot{\mathbf{u}}_t + \mathbf{K}_t\Delta\mathbf{u}_t = \tilde{\mathbf{R}}_{t+\tau} - \mathbf{M}\ddot{\mathbf{u}}_t - \mathbf{C}\dot{\mathbf{u}}_t - \mathbf{F}_t^e \tag{9*}$$

where $\tilde{\mathbf{R}}_{t+\tau}$ is a 'projected' load equal to $\mathbf{R}_t + \theta(\mathbf{R}_{t+\Delta t} - \mathbf{R}_t)$. Equation (3*) is solved for $\mathbf{u}_{t+\tau}$ by simply substituting for $\ddot{\mathbf{u}}_{t+\tau}$ and $\dot{\mathbf{u}}_{t+\tau}$ using equations (15) and (16). To solve for $\mathbf{u}_{t+\tau}$ from equation (9*), we further use the relationships $\Delta\mathbf{u}_t = \mathbf{u}_{t+\tau} - \mathbf{u}_t$, $\Delta\dot{\mathbf{u}}_t = \dot{\mathbf{u}}_{t+\tau} - \dot{\mathbf{u}}_t$ and $\Delta\ddot{\mathbf{u}}_t = \ddot{\mathbf{u}}_{t+\tau} - \ddot{\mathbf{u}}_t$. With $\mathbf{u}_{t+\tau}$ known the accelerations and velocities at time $t+\tau$ are obtained using again equations (15) and (16).

At the desired time $t+\Delta t$ the required accelerations, velocities and displacements are given by the linear acceleration assumption:

$$\ddot{\mathbf{u}}_{t+\Delta t} = \left(1 - \frac{1}{\theta}\right)\ddot{\mathbf{u}}_t + \frac{1}{\theta}\ddot{\mathbf{u}}_{t+\tau} \tag{17}$$

$$\dot{\mathbf{u}}_{t+\Delta t} = \dot{\mathbf{u}}_t + \frac{\Delta t}{2}(\ddot{\mathbf{u}}_t + \ddot{\mathbf{u}}_{t+\Delta t}) \tag{18}$$

$$\mathbf{u}_{t+\Delta t} = \mathbf{u}_t + \Delta t\dot{\mathbf{u}}_t + \frac{\Delta t^2}{6}(\ddot{\mathbf{u}}_{t+\Delta t} + 2\ddot{\mathbf{u}}_t) \tag{19}$$

An efficient computer oriented formulation of the step-by-step analysis of linear systems is given in Table I. In order to minimize computer storage the damping matrix is assumed to be a linear combination of the stiffness and mass matrix. Also, the equations are solved for a vector $\mathbf{u}_t^*$ which does not have physical significance. However, the use of this vector eliminates the need to store the original stiffness matrix $\mathbf{K}$ during the solution procedure.

Table I. Summary of step-by-step algorithm for linear structural systems

Initial calculations

1. Form stiffness matrix **K** and mass matrix **M**.
2. Calculate the following constants (assume $\mathbf{C} = \alpha\mathbf{M}+\beta\mathbf{K}$):

$$\tau = \theta\Delta t, \quad \theta \geqslant 1{\cdot}37 \qquad b_6 = 2+\frac{\tau}{2}b_3$$

$$b_0 = \left(1+\frac{3}{\tau}\beta\right) \qquad b_7 = \frac{6}{\theta\tau^2 b_0}$$

$$b_1 = \left(\frac{6}{\tau^2}+\frac{3}{\tau}\alpha\right) \qquad b_8 = \frac{3}{\tau}\beta b_7-\frac{6}{\theta\tau^2}$$

$$b_2 = \frac{b_1}{b_0} \qquad b_9 = 2\beta b_7-\frac{6}{\theta\tau}$$

$$b_3 = \alpha-\beta b_2 \qquad b_{10} = 1-\frac{3}{\theta}+\frac{\tau}{2}\beta b_7$$

$$b_4 = \frac{6}{\tau^2}+\frac{3}{\tau}b_3 \qquad b_{11} = \frac{\Delta t}{2}$$

$$b_5 = \frac{6}{\tau}+2b_3, \qquad b_{12} = \frac{\Delta t^2}{6}$$

3. Form effective stiffness matrix $\mathbf{K}^* = \mathbf{K}+b_2\mathbf{M}$.
4. Triangularize $\mathbf{K}^*$.

For each time increment

1. Form effective load vector $\mathbf{R}^*$
$$\mathbf{R}_t^* = \mathbf{R}_t+\theta(\mathbf{R}_{t+\Delta t}-\mathbf{R}_t)+\mathbf{M}[b_4\mathbf{u}_t+b_5\dot{\mathbf{u}}_t+b_6\ddot{\mathbf{u}}_t]$$
2. Solve for effective displacement vector $\mathbf{u}_t^*$
$$\mathbf{K}^*\mathbf{u}_t^* = \mathbf{R}_t^*$$
3. Calculate new acceleration, velocity and displacement vectors,
$$\ddot{\mathbf{u}}_{t+\Delta t} = b_7\mathbf{u}_t^*+b_8\mathbf{u}_t+b_9\dot{\mathbf{u}}_t+b_{10}\ddot{\mathbf{u}}_t$$
$$\dot{\mathbf{u}}_{t+\Delta t} = \dot{\mathbf{u}}_t+b_{11}(\ddot{\mathbf{u}}_{t+\Delta t}+\ddot{\mathbf{u}}_t)$$
$$\mathbf{u}_{t+\Delta t} = \mathbf{u}_t+\Delta t\dot{\mathbf{u}}_t+b_{12}(\ddot{\mathbf{u}}_{t+\Delta t}+2\ddot{\mathbf{u}}_t)$$
4. Calculate element stresses if desired.
5. Repeat for next time increment.

The algorithm for the dynamic analysis of structural systems with physical or geometrical nonlinearities is summarized in Table II.

Stability and accuracy of the step-by-step integration

It is most important that the integration method be unconditionally stable for general application. This essentially means that a bounded solution is obtained for any size of time step Δt. A conditionally stable scheme requires for a bounded solution a time step smaller than a certain limit. Naturally, the accuracy of the solution always depends on the size of the time step, but using an unconditionally stable scheme the time step is chosen with regard to accuracy only and not with regard to stability. This generally allows a much larger time step to be used.

A stability analysis of the integration method shows that it is unconditionally stable provided $\theta \geqslant 1{\cdot}37$.

In order to obtain an idea of the accuracy which can be obtained in the numerical integration we consider the analysis of a linear system with n degrees of freedom. The equations governing free vibration conditions with damping ignored are

$$\mathbf{M}\ddot{\mathbf{u}}+\mathbf{K}\mathbf{u} = 0 \qquad (20)$$

Table II. Summary of step-by-step algorithm for nonlinear structural systems

Initial calculations

1. Form stiffness matrix **K** and mass matrix **M**.
2. Solve for initial displacements, strains, stresses and internal forces due to static loads.
3. Calculate the following constants:

$$\tau = \theta\Delta t, \quad \theta \leqslant 1{\cdot}37$$

$$a_0 = 6/\tau^2 \quad a_4 = a_0/\theta \quad a_7 = \Delta t/2$$

$$a_1 = 3/\tau \quad a_5 = -a_2/\theta \quad a_8 = \Delta t^2/6$$

$$a_2 = 2a_1 \quad a_6 = 1-3/\theta$$

$$a_3 = \tau/2$$

For each time increment

1. Calculate tangent stiffness matrix $\mathbf{K}_t$.
2. Form effective stiffness matrix $\mathbf{K}_t^* = \mathbf{K}_t + a_0\,\mathbf{M} + a_1\,\mathbf{C}$.
3. Triangularize $\mathbf{K}_t^*$.
4. Form effective load vector $\mathbf{R}_t^*$

$$\mathbf{R}_t^* = \mathbf{R}_t + \theta(\mathbf{R}_{t+\Delta t} - \mathbf{R}_t) - \mathbf{F}_t + \mathbf{M}(a_2\,\dot{\mathbf{u}}_t + 2\ddot{\mathbf{u}}_t) + \mathbf{C}(2\dot{\mathbf{u}}_t + a_3\,\ddot{\mathbf{u}}_t)$$

5. Solve for incremental displacement vector $\Delta\mathbf{u}_t$

$$\mathbf{K}_t^*\,\Delta\mathbf{u}_t = \mathbf{R}_t^*$$

6. Calculate new acceleration, velocity and displacement vectors:

$$\ddot{\mathbf{u}}_{t+\Delta t} = a_4\,\Delta\mathbf{u}_t + a_5\,\dot{\mathbf{u}}_t + a_6\,\ddot{\mathbf{u}}_t$$

$$\dot{\mathbf{u}}_{t+\Delta t} = \dot{\mathbf{u}}_t + a_7(\ddot{\mathbf{u}}_{t+\Delta t} + \ddot{\mathbf{u}}_t)$$

$$\mathbf{u}_{t+\Delta t} = \mathbf{u}_t + \Delta t\dot{\mathbf{u}}_t + a_8(\ddot{\mathbf{u}}_{t+\Delta t} + 2\ddot{\mathbf{u}}_t)$$

7. Calculate strains, stresses, and internal force vector $\mathbf{F}_{t+\Delta t}$.
8. For next step return to 1 or 4.

Let the matrix $\boldsymbol{\phi}$ contain the n eigenvectors of the system, then equation (20) can be transformed into n uncoupled equations

$$\ddot{\mathbf{X}} + \boldsymbol{\Omega}^2\mathbf{X} = 0 \tag{21}$$

where $\mathbf{u} = \boldsymbol{\phi}\mathbf{X}$, $\boldsymbol{\Omega}^2 = \text{diag}(\omega_i^2)$, $\omega_i = 2\pi/T_i$ and the T_i are the natural periods of the system. It is obvious that the integration of equation (20) is equivalent to the integration of equation (21). But the advantage of using equation (21) is that the accuracy which is obtained in the integration of this equation can be assessed by studying the accuracy which is obtained in the analysis of a single degree of freedom system.

As an example, Figures 1 and 2 show the errors associated with the solution of the initial value problem indicated in the figures as a function of $\Delta t/T$ and θ, where T is the natural period of the single degree of freedom system. The numerical errors are conveniently measured as a percentage period elongation and amplitude decay. It is seen that for $\Delta t/T$ smaller than about 0·01 the numerical error is small; but for $\Delta t/T > 0{\cdot}2$ the amplitude decay is very large. Therefore, in the solution of equation (21) with equivalent initial conditions the vibration modes with periods smaller than about $5\Delta t$ may be said to be filtered out of the solution.

These observations about the numerical integration errors are quite general, although only the solution of one particular initial value problem was presented.[12] The observations can be used in the selection of an appropriate time step size in a practical analysis of a linear or 'slightly' nonlinear system.

Selection of time step size

In the dynamic analysis of most structures only frequencies in a specified range are of practical interest. In general the type of loading defines which frequencies are significant, and how small a time step should be used.

For example in the case of earthquake loading, in which excitation components with periods smaller than about 0·05 s generally are not accurately recorded, there is very little justification to include the response in these higher frequencies in the analysis. Figures 1 and 2 can be used as a guide to select a time step Δt which produces an acceptable integration error in the low mode response and filters out the higher mode response.

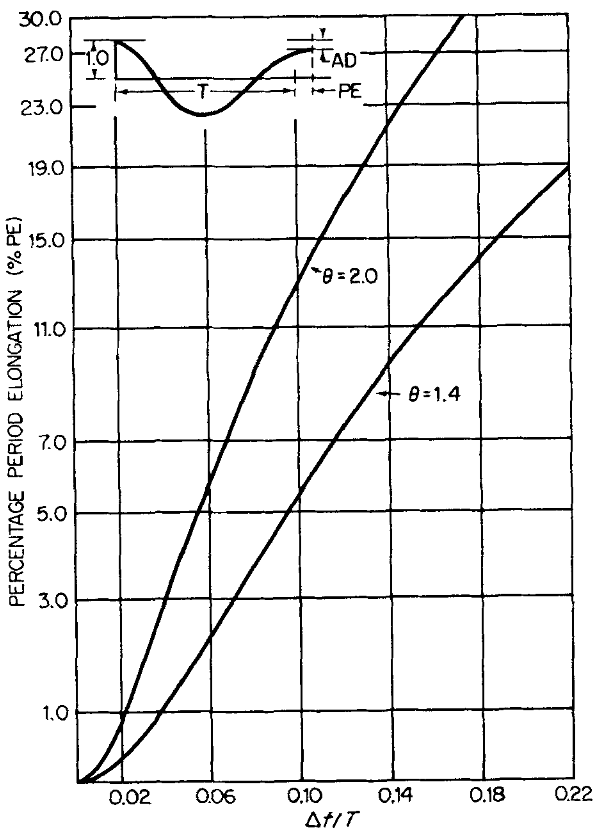

Figure 1. Percentage period elongation as a function of $\Delta t/T$

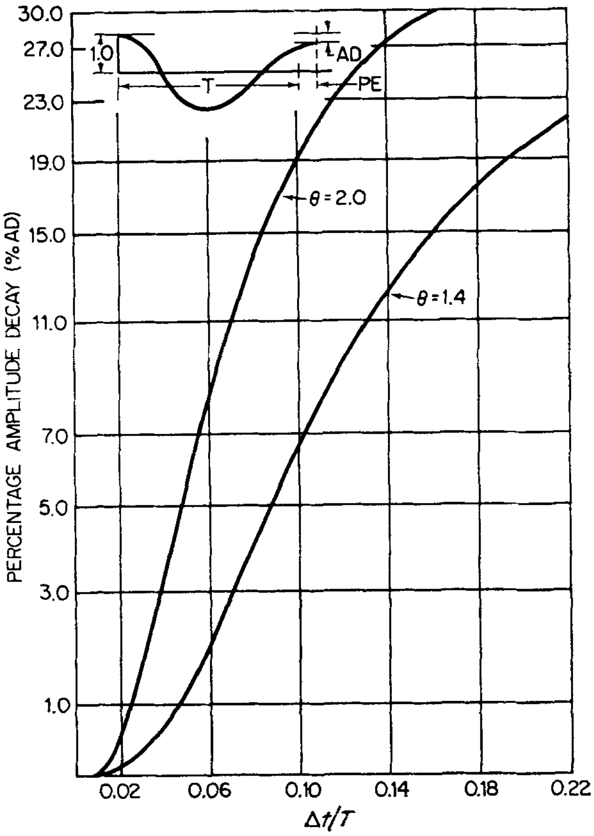

Figure 2. Percentage amplitude decay as a function of $\Delta t/T$

In general, to select an appropriate integration time step for a given problem, it is necessary first to evaluate the frequency components of the loading which can be predicted accurately. Next, the form of the finite element idealization must be selected in order to define accurately these frequencies. Particular attention needs to be given to the fact that the highest frequencies of the lumped parameter system are always in error when compared to the continuous problem. Finally, a time step must be selected which accurately represents the frequency components in the load and which suppresses the higher frequencies of the lumped parameter system. The period elongations and amplitude decays resulting from the numerical integration should be small compared with the physical damping which exists in the real material for all frequencies of significance.

EXAMPLES

Linear undamped forced vibration of a cantilever beam

A demonstration of the effectiveness of the numerical method presented in this paper is provided by a linear dynamic analysis of a cantilever beam. The finite element idealization of the beam and the time variation of the load applied at point A, are shown in Figures 3 and 4. Figure 5 gives the time variation of the normalized displacement of the beam tip. It is shown that the well known linear acceleration method, which is conditionally stable, fails to yield a bounded response. But the integration method presented in the paper approximates the 'exact' solution well for the relatively large $\Delta t/T$ ratio chosen. As expected, the accuracy of the method is better for the smaller value of θ.

Settlement of soil under dynamic pressure

The dynamic analysis of a confined cylindrical nonlinear soil sample, with dimensions 3 in. by 6 in. as shown in Figure 6, is considered next. The time varying pressure, shown in Figure 7, was applied at the top of the sample, and the time step used in the numerical integration was 0·005 s.

The stress–strain relationship of the soil is nonlinear and divided into bulk and shear parts. It can be shown that given the initial shear modulus, the only relation required for approximately defining the nonlinear hysteretic behaviour of soil under dynamic pressure is the hydrostatic pressure–volume change diagram. For the soil in this example the pressure–volume relation is given in Figure 8. The results of a linear and two

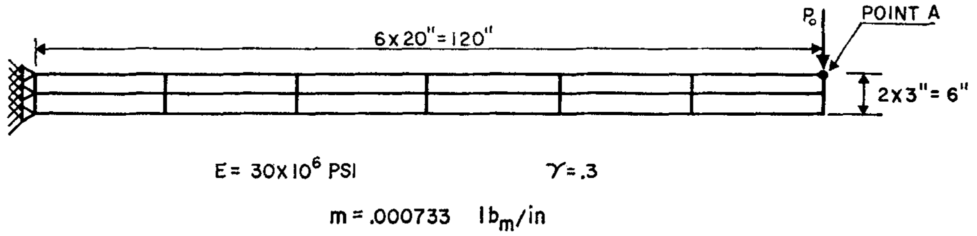

Figure 3. Finite element idealization of a cantilever beam

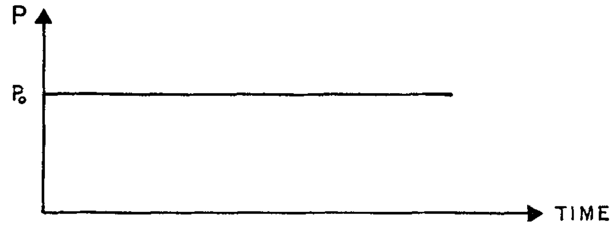

Figure 4. Time variation of the load

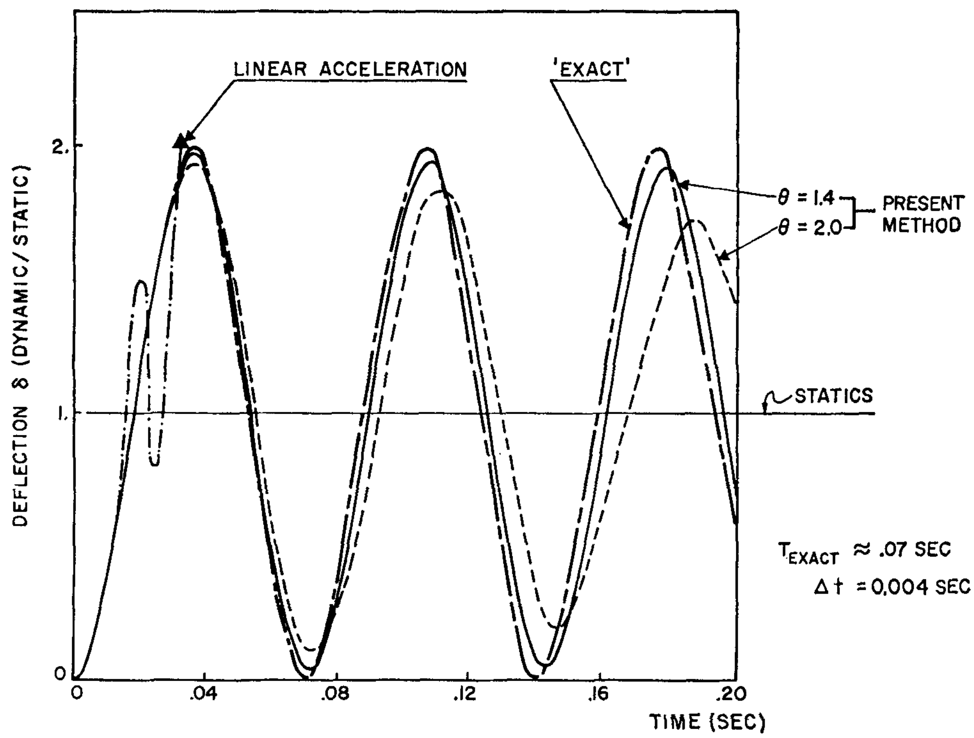

Figure 5. Deflection of a cantilever beam

nonlinear finite element analyses together with experimental data are given in Figure 9. For this example it is difficult to obtain the same results in analysis and experiment due to the various types of approximations and errors involved in both procedures. However, the nonlinear analyses capture the main behaviour of the specimen which is a finite settlement. The linear analysis shows undesirable behaviour since the model cannot permit permanent strain. Note that for $\theta = 2{\cdot}0$ the response is smoother, but when $\theta = 1{\cdot}5$ the results from analysis and experiment are closer.

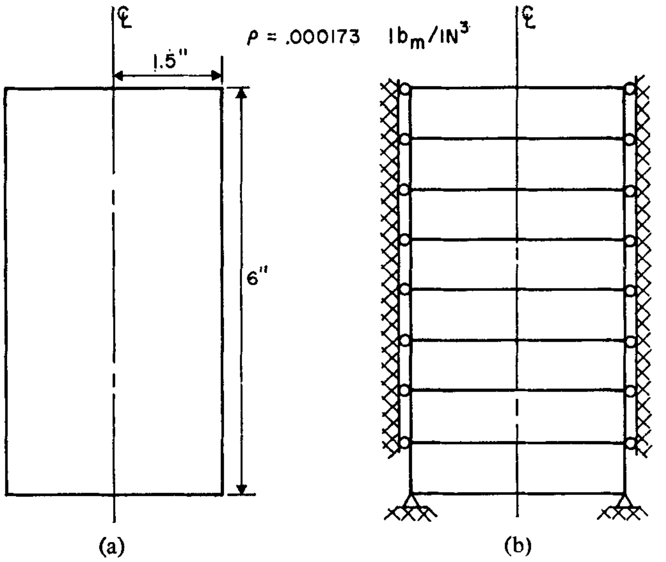

Figure 6. (a) Soil sample; (b) Finite element model

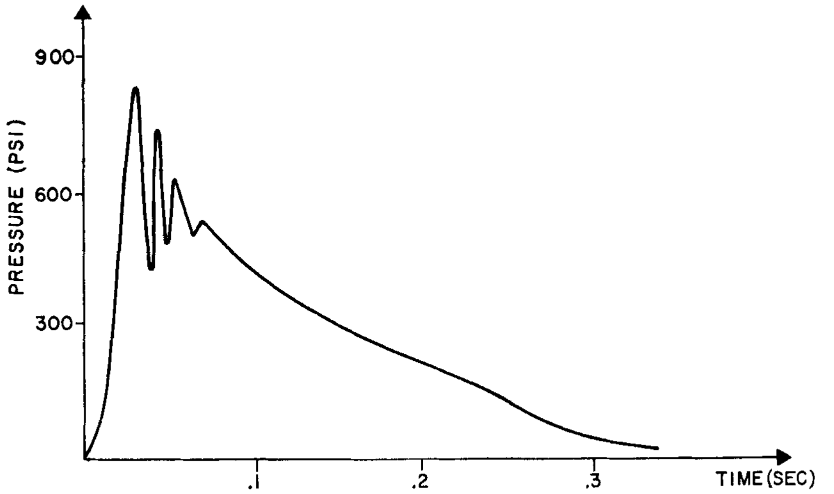

Figure 7. Time variation of blast pressure

Dynamic snap-through of an arch

The last example is a dynamic large displacement analysis of a circular arch subjected to a uniform time varying pressure. The finite element and load idealizations are shown in Figures 10 and 11 respectively. The following data have been used in the example:

$\beta = 30°$ $\rho = 0{\cdot}00625\ \mathrm{lb_m/in^3}$

$h = 2{\cdot}0$ in. $E = 3326{\cdot}2\ \mathrm{lb/in^2}$

$R = 72{\cdot}95$ in. $\Delta t = 0{\cdot}025$, $0{\cdot}0125$ and $0{\cdot}00625$ s

Figure 12 shows a comparison between the results obtained by Humphreys[13] and results obtained in the analysis of the finite element assemblage using the integration technique presented in the paper. Three analyses with time steps equal to 0·025, $\frac{1}{2}$(0·025) and $\frac{1}{4}$(0·025) were carried out. These indicate convergence of the step-by-step integration analysis of the finite element assemblage as Δt becomes small. The remaining

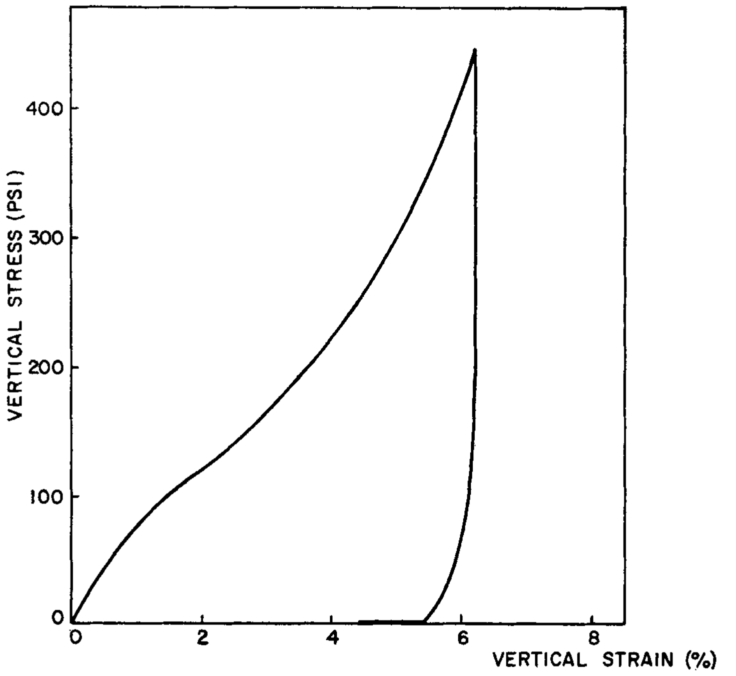

Figure 8. Stress–strain diagram for soil

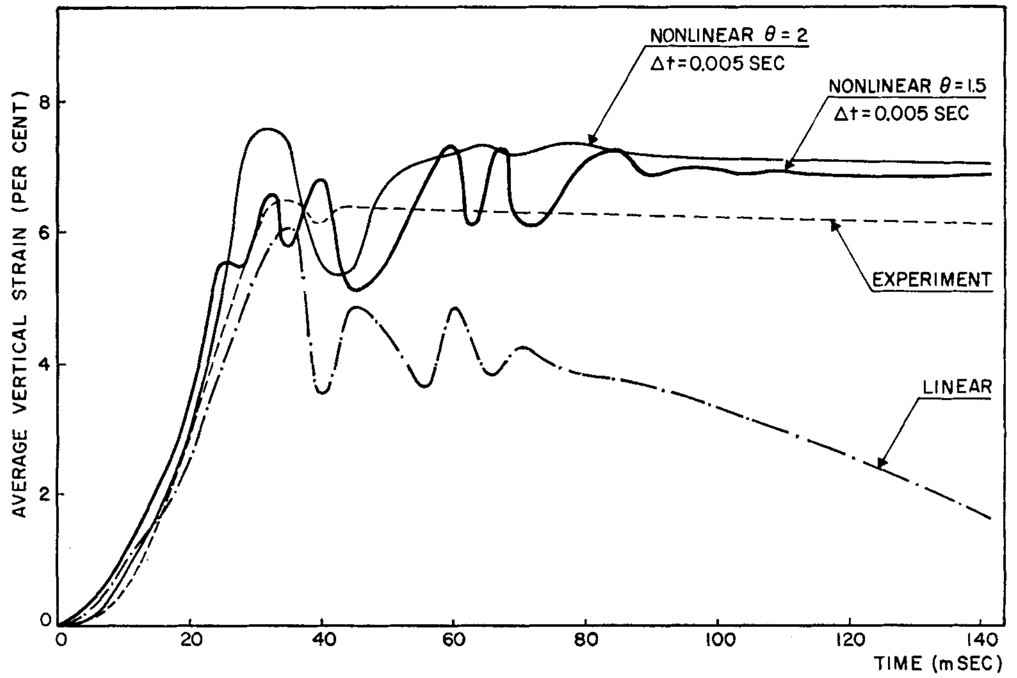

Figure 9. Dynamic response of soil sample

discrepancy between Humphreys' series solution and the finite element analysis arises from other approximations involved in both procedures. Note, for example, that Humphreys truncates the series solution at a finite number of terms, and that the geometry of the arch is not modelled exactly in the finite element analysis.

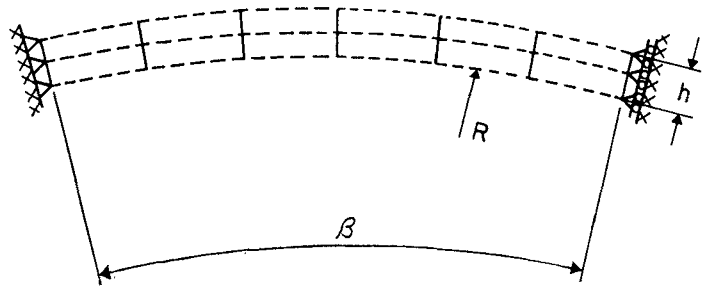

Figure 10. Finite element idealization of arch

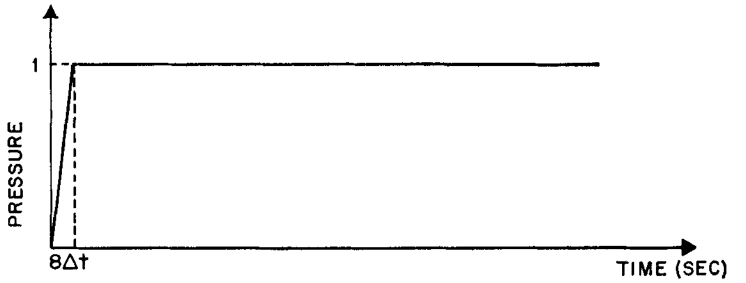

Figure 11. Load idealization

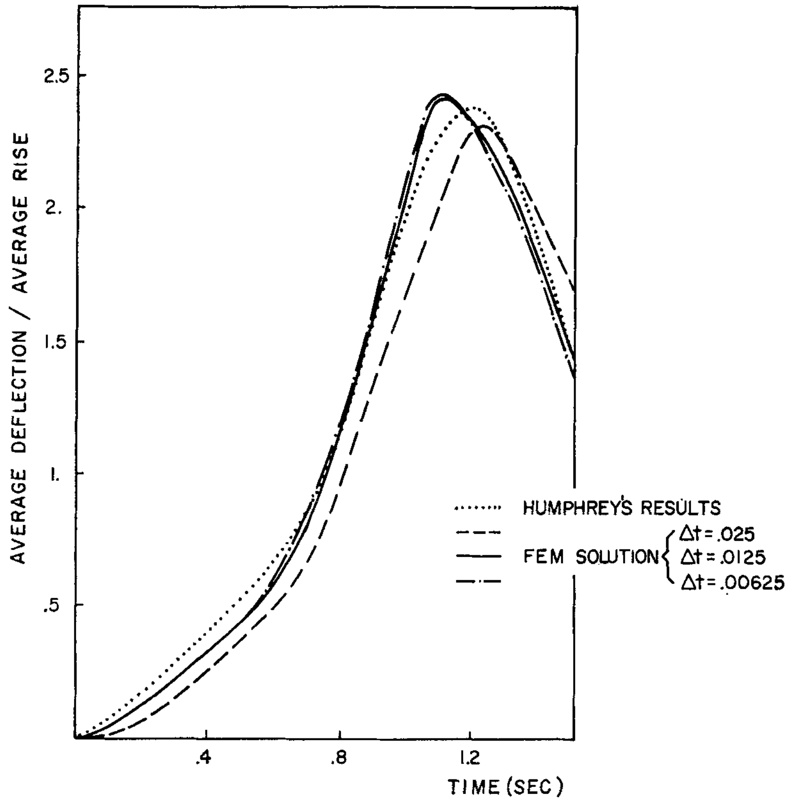

Figure 12. Deflection of arch versus time

CONCLUSIONS

An efficient solution technique for the nonlinear static and dynamic analysis of an assemblage of structural elements has been presented. In the solution, the nonlinear equations of motion are written in an incremental form, which lends itself to a check of dynamic equilibrium in the deformed configuration. The equations are integrated using an unconditionally stable scheme. To obtain an idea of the accuracy of the numerical integration a linear system was considered. This led to the presentation of practical guidelines for the selection of the time step size for different types of dynamic loadings. Finally, the example analyses showed the effectiveness of the solution technique.

Although this solution technique can already be used in many different practical analyses, more research is required in important problem areas. A most difficult problem in many nonlinear analyses is the evaluation of appropriate tangent mass, damping and stiffness matrices. The numerical solution given here assumes that the linearization over the time increment is adequate and that at most a few iterations will reduce the residual force vector in equation (8) to a negligible quantity. Unless this dynamic equilibrium condition is satisfied, we cannot expect any accuracy in the solution.

Another area in which more research is required is the evaluation of better estimates for the solution accuracy. The guidelines given in the paper for the selection of the time step size can only be used when systems with small nonlinearities are analysed. If the system is strongly nonlinear, for example, because sudden changes in the material properties occur, a much smaller time step may be necessary. At present, in most analyses the time step increment is determined by trial and error, which can be a very expensive process.

REFERENCES

1. A. H. S. Ang, 'Numerical approach for wave motions in nonlinear solid media', *Proc. Conf. on Matrix Methods in Struct. Mech., Dayton, Ohio*, 1965, AFFDL–TR–68–80, 1966.
2. M. Khojasteh-Bakht, 'Analysis of elastic-plastic shells of revolution by the finite element method', *SESM Report* 67–1, *University of California, Berkeley*, 1967.
3. M. Dibaj, 'Nonlinear seismic response of earth structures', *Ph.D. dissertation, University of California, Berkeley*, 1969.
4. H. C. Martin, 'Large deflection and stability analysis by the direct stiffness method', *Jet Prop. Lab., Tech. Report, Pasadena*, 32–931, 1965.
5. D. W. Murrary, 'Large deflection analysis of plates', *SESM Report* 67–44, *University of California, Berkeley*, 1967.
6. S. Yaghmai, 'Incremental analysis of large deformation in mechanics of solids with application to axisymmetric shells of revolution', *SESM Report* 68–17, *University of California, Berkeley*, 1968.
7. R. W. Clough and E. L. Wilson, 'Dynamic finite element analysis of arbitrary thin shells,' *Computers and Structures*, **1**, 33–56 (1971).
8. N. M. Newmark, 'A method of computation for structural dynamics', *Proc. ASCE*, **85**, No. EM3, 67–94 (1959).
9. C. W. Gear, 'The automatic integration of stiff ordinary differential equations', *IFIP Congress 68, Edinburgh*, 1968.
10. E. L. Wilson and J. Penzien, 'Evaluation of orthogonal damping matrices', *Int. J. Num. Meth. Engng*, **4**, 5–10 (1972).
11. I. Farhoomand, 'Nonlinear dynamic stress analysis of two-dimensional solids', *Ph.D. dissertation, University of California, Berkeley*, 1970.
12. K. J. Bathe and E. L. Wilson, 'Stability and accuracy analysis of direct integration methods', submitted for publication.
13. J. S. Humphreys, 'On dynamic snap buckling of shallow arches', *AIAA J.*, **4**, 878–886 (1966).